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An $Sl(2,\mathbb{Z})$ Multiplet of Nine-Dimensional Type II Supergravity Theories

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Abstract

We show that only by performing generalized dimensional reductions all possible brane configurations are taken into account and one gets the complete lower-dimensional theory. We apply this idea to the reduction of type IIB supergravity in an $SL(2,\mathbb{R})$-covariant way and establish T duality for the type II superstring effective action in the context of generalized dimensional reduction giving the corresponding generalized Buscher's T duality rules.

The full (generalized) dimensional reduction involves all the S duals of D-7-branes: Q-7-branes and a sort of composite 7-branes. The three species constitute an $SL(2,\mathbb{Z})$ triplet. Their presence induces the appearance of the triplet of masses of the 9-dimensional theory.

The T duals, including a "KK-8A-brane", which must have a compact transverse dimension have to be considered in the type IIA side. Compactification of 11-dimensional KK-9M-branes (a.k.a. M-9-branes) on the compact transverse dimension give D-8-branes while compactification on a worldvolume dimension gives KK-8A-branes. The presence of these KK-monopole-type objects breaks translation invariance and two of them given rise to an $SL(2,\mathbb{R})$-covariant *massive 11-dimensional supergravity* whose reduction gives the massive 9-dimensional type II theories.

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1 Introduction

During the last few years, the study of the low-energy string effective action has shown itself to be most profitable. It has helped us to establish duality between many pairs of string theories (duality of the effective actions being a necessary condition) and has provided us with semiclassical solutions describing the long-range fields of perturbative and non-perturbative states of string theories. In particular, the now popular relation between M theory and type IIA string theory was first suggested by the relation between 11-dimensional and type IIA supergravity [1]. 11-dimensional supergravity is our best source of knowledge about M theory. On the other hand, in the stringy microscopic explanation of the origin of black-hole entropy (see e.g. Ref. [2]) it is fundamental to have a semiclassical solution describing the black hole to calculate the area of the horizon.

Superstring effective actions are nothing but the actions of supergravity theories. Thus, there is much to be learned from the old techniques used to compactify them if we translate them to string language. For instance, compactifying the action of $N = 1, d = 10$ supergravity on a circle, it is easy to recover Buscher's T duality rules [3] as a global discrete symmetry of the 9-dimensional theory that interchanges two vector fields associated to momentum modes' charges and winding modes' charges [4, 5]. In the type II context, only through the use of the effective action it was possible to derive the generalization of Buscher's T duality rules [6].

In Ref. [7] Scherk and Schwarz proposed the method of "generalized dimensional reduction": In standard dimensional reduction it is required that all fields are independent of the coordinates of the compact dimensions. However, when there are global symmetries (always present in the internal dimensions), in order to guarantee that the lower-dimensional theory is independent of them, it is enough to require that the fields depend on them in a certain way. The terms depending on the internal directions generate mass terms on lower dimensions.

The authors of Ref. [8] first used the idea of Scherk-Schwarz generalized dimensional reduction [7] with global symmetries of no (known) geometrical origin. They applied it to the symmetry under constant shifts of the type IIB RR scalar and obtained a massive 9-dimensional type II theory, precisely the one one obtains through standard dimensional reduction from Romans' massive type IIA supergravity [9]. Another important result of Ref. [8] is that they identified the presence of D-7-branes in the background as the origin of the 9-dimensional mass. By T duality arguments the mass parameter of Romans' theory was identified with the presence of D-8-branes[3], (more precisely as a "(-1)-form RR potential") and Buscher-type T duality rules could then be derived.

In this theory $SL(2,\mathbb{R})$ (the classical S duality group[4]) was broken. This was to be expected since the S duals of D-7-branes were not present. In other words, the global

[3]This identification had already been made in Ref. [10].

[4]$SL(2,\mathbb{R})$ is broken to $SL(2,\mathbb{Z})$ by quantum effects such as charge quantization. Most of our considerations throughout this paper will be purely classical and thus we will mostly talk about $SL(2,\mathbb{R})$. However, at specific places the restriction to the discrete $SL(2,\mathbb{Z})$ will be important and we will deal with it in full detail.

symmetry chosen was only a subgroup of the full $SL(2,\mathbb{R})$ global symmetry available.

In Ref. [11] a systematic way of performing the generalized dimensional reduction associated to the type IIB S duality group was proposed. Again, the full S duality group was not used and the dimensional reduction was not finished due to the complications introduced by the self-dual 5-form. The resulting 9-dimensional massive type II theory was then incomplete and not $SL(2,\mathbb{R})$-invariant.

In fact, the breaking of S duality seems unavoidable. However, S duality is supposed to be an exact symmetry of type IIB superstring theory. The solution to this puzzle is that one has to take into account the transformation of the mass parameters, which should be considered "(-1)-form" potentials, and include all the S duality related mass parameters. Then, the method of Ref. [11] suggests that, by making use of the full $SL(2,\mathbb{R})$ group in the generalized dimensional reduction a 9-dimensional massive $SL(2,\mathbb{R})$-*covariant* type II theory should be obtainable. If the mass parameters are considered fixed constants of the theory, then what one obtains can be considered as an $SL(2,\mathbb{R})$ multiplet of 9-dimensional massive type II theories.

The first goal of this paper is to find this theory and interpret it in terms of 10-dimensional 7-branes. We also want to get a better understanding of the method of generalized dimensional reduction and we will propose two alternative methods giving the same results.

Since (as we will argue after we present a toy model of generalized dimensional reduction in Section 1.1) this 9-dimensional theory should be considered *the* 9-dimensional type II theory[5], one expects T duality to hold in this context.

Our second goal in this paper will be to establish T duality in the context of generalized dimensional reduction of the type IIB theory. To achieve this goal one faces an important problem that can be expressed in two different ways:

1. It is clear, from the above discussion that one has to identify the S duals of the D-7-brane and then one has to identify their T duals. The T dual theory will be type IIA theory in a background containing these objects.

2. One has to find the generalization of type IIA supergravity which gives rise to the 9-dimensional masses we get from the type IIB side. However it does not seem possible to further generalize Romans' massive type IIA supergravity.

A clue to the resolution of this problem is the fact that the type IIB $SL(2,\mathbb{R})$ symmetry is identical to the $SL(2,\mathbb{R})$ symmetry of 11-dimensional supergravity compactified on T^2 which acts on the internal manifold [6, 12, 13, 14]. S duality then interchanges the 11-dimensional theory that gives rise to Romans' theory with the 10-dimensional theory associated to the T duals of the S duals of D-7-branes.

The 11-dimensional origin of Romans' theory is somewhat mysterious because 11-dimensional supergravity cannot be deformed to include a mass or a cosmological constant

[5]A slightly more general massive 9-dimensional type II theory may be constructed, though. This will be explained in the Conclusion Section.

according to the no-go theorem of Refs. [15]. In Refs. [16, 17, 18], though, a modification of 11-dimensional supergravity that breaks 11-dimensional covariance (one of the hypothesis of the no-go theorem) was proposed. This theory gives Romans' type IIA upon dimensional reduction. The original motivation for the construction of this theory was the realization that the worldvolume effective actions of the extended solitons of Romans' type IIA can be derived from 11-dimensional gauged σ-models[6]. The gauging of an isometry breaks the gauge invariance of the Wess-Zumino term. To restore it, it is not enough to modify this term. One has to modify the gauge transformations o the 11-dimensional fields. The massive 11-dimensional theory constructed in Ref. [18] is precisely the one which is invariant under these modified gauge transformations.

The reason for the explicit breaking of 11-dimensional covariance was not sufficiently explained. It has been suggested recently in Ref. [21] that it is due to the presence of some objects, M-9-branes which we call KK-9M-branes. Now we are saying that, due to S duality, a 10-dimensional type IIA theory with broken Poincaré covariance is needed in order to make contact with the general massive 9-dimensional type II theory that we are about to construct. Furthermore, our previous arguments show that this breaking of covariance is due to the presence of certain objects: The T duals of the S duals of D-7-branes, which we call KK-8A-branes and which can be obtained by dimensional reduction of KK-9M-branes. This theory can be obtained by the compactification of the massive 11-dimensional supergravity of Ref. [18] along a different coordinate.

These objects must be somewhat similar to type IIA Kaluza-Klein (KK) monopoles, which are the T dual of the S dual of D-5-branes (solitonic NS-NS S-5-branes): A dimension transverse to their worldvolume has to be compactified on a circle and there is an isometry associated to it. The presence of this isometry is the reason for the breaking of 10-dimensional (and 11-dimensional) covariance. Compactification of the KK-9M-brane along a worldvolume direction gives the 10-dimensional type IIA KK-8A-brane while compactification along the isometry direction gives the D-8-brane [21]. Compactification of the KK-8A-brane along the isometry gives a 9-dimensional Q-7-brane, the S dual of a 9-dimensional D-7-brane. This is similar to what happens with KK-monopoles: In eleven dimensions the KK-monopole can be called KK-7M-brane. Compactification along a worldvolume direction gives the type IIA KK-monopole that we can call KK-6A-brane and compactification along the isometry direction gives the D-6-brane. Further compactification of the KK-6A-brane along the isometry gives a 9-dimensional NS-NS S-5-brane. These, and more relations, are depicted in Figure 4 and will be explored in Section 6.

The worldvolume theory of the KK-8A- and the KK-9M-brane must also be given by a gauged σ-model[7], where the symmetry gauged is associated to the isometry that these objects must have in the compact dimension, just as happens with the usual KK monopoles [19].

Thus, we are lead to the following picture which solves our problem: There is a massive

[6]The effective action of the Kaluza-Klein (KK) monopole is also a gauged σ-model [19]. However, in this case the gauging is not associated to any mass parameter. Actually, to describe the KK in massive background the gauging of a second isometry is necessary [20].

[7]The corresponding action for the KK-9M-brane as been written in Ref. [21].

11-dimensional supergravity theory associated to the presence of *two* KK-9M-branes and thus has 11-dimensional covariance broken in two directions: the isometries of the KK-9M-branes. This theory has $SL(2,\mathbb{R})$-covariance in those two special isometric directions. One can eliminate one isometry by compactifying along it obtaining a modification of Romans' type IIA theory with covariance broken in the other isometry direction. This is type IIA in the presence of a D-8-brane and a KK-8-brane. Reducing further along the other isometry direction gives the desired 9-dimensional massive type II theory, which is type II theory in presence of D-7-branes and and their S dual Q-7-branes.

We will show that this is the right picture and we will comment on its possible generalizations in the Conclusion.

At this point it is perhaps convenient to study a simple example to illustrate some of our ideas.

1.1 Generalized Dimensional Reduction of the Einstein-Dilaton Theory

We consider the following toy model which exhibits the general features of generalized dimensional reduction associated to global symmetries with no geometrical origin[8]:

$$\hat{S} = \int d^d\hat{x}\sqrt{|\hat{g}|}\ \left[\hat{R} + \tfrac{1}{2}\left(\partial\hat{\phi}\right)^2\right] . \tag{1.1}$$

This action is invariant under constant shifts of the scalar $\hat{\phi}$, the reason being that $\hat{\phi}$ only occurs through its derivatives. The presence of this global symmetry allows us to extend the general Kaluza-Klein Ansatz (i.e. all fields, and in particular $\hat{\phi}$, are independent of some coordinate, say z) to a more general Ansatz in which $\hat{\phi}$ depends on z in a particular way:

$$\hat{\phi}(x,z) = \hat{\phi}^{\mathrm{b}}(x) + mz\,, \qquad \hat{x}^{\hat{\mu}} = (x^{\mu}, z)\,, \tag{1.2}$$

where the superscript ^{b} stands for *bare*, or z-independent.

This dependence on z can be produced by a *local* shift of $\hat{\phi}(x)$ with a parameter linear in z. The invariance of the action under constant shifts ensures that the action will not depend on z.

This is only a practical recipe to write a good Ansatz. To understand better what one is doing, one has to recall that z is a coordinate on a circle S^1 subject to the identification $z \sim z + 2\pi l$. In standard Kaluza-Klein reduction one only considers single-valued fields, so that the needed Fourier decomposition of the fields living on $\mathcal{M}\otimes S^1$, reads

$$\hat{\phi}\,(\hat{x}) \;=\; \sum_{n\in\mathbb{Z}} e^{2\pi n z/l}\,\phi^{(n)}(x)\;. \tag{1.3}$$

[8]In this section we use hats for d-dimensional objects and no hats for $(d-1)$-dimensional objects.

Dimensional reduction then means keeping the massless modes, *i.e.* $\phi^{(0)}$, only. Some fields can be multivalued, however. If the scalar $\hat{\phi}$ is such that $\hat{\phi} = \hat{\phi} + 2\pi m$, the above Fourier expansion is enhanced to

$$\hat{\phi}\,(\hat{x}) \;=\; \frac{mNz}{l} + \sum_{n\in\mathbb{Z}} e^{2\pi n z/l}\,\phi^{(n)}(x)\;, \tag{1.4}$$

where $N \in \mathbb{Z}$ labels the different *topological sectors.* Now, the action for a field living on an S^1 is always invariant under arbitrary shifts of the field, even if the field is to be identified under discrete shifts. This then ensures that the lower dimensional theory does not depend on z, the dimensional reduction, if only $\frac{mNz}{l} + \phi^{(0)}$ is kept. Each topological sector is characterized by the charge

$$N \;=\; \lim_{x\to\infty} \frac{1}{2\pi l m} \oint d\hat{\phi}\;, \tag{1.5}$$

which is nothing but the *winding number.*

A more physical interpretation of the technical description of the generalized dimensional reduction recipe will be given later on.

Making use of the standard KK Ansatz for the Vielbein

$$\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) = \begin{pmatrix} e_\mu{}^a & kA_{(1)\;\mu} \\ 0 & k \end{pmatrix}\,, \qquad \left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) = \begin{pmatrix} e_a{}^\mu & -A_{(1)\;a} \\ 0 & k^{-1} \end{pmatrix}\,, \tag{1.6}$$

we readily obtain the $(d-1)$-dimensional action

$$S = \int d^{d-1}x\sqrt{|g|}\;k\;\left[R - \tfrac{1}{4}k^2 F_{(2)}^2 + \tfrac{1}{2}\left(\mathcal{D}\phi\right)^2 - \tfrac{1}{2}m^2 k^{-2}\right]\,, \tag{1.7}$$

where the field strengths are defined by

$$\begin{cases} F_{(2)\;\mu\nu} \;=\; 2\partial_{[\mu}A_{(1)\;\nu]}\;, \\ \mathcal{D}_\mu\phi \;=\; \partial_\mu\phi - mA_{(1)\;\mu}\;, \end{cases} \tag{1.8}$$

and

$$\phi \equiv \hat{\phi}^{\mathrm{b}}\,. \tag{1.9}$$

A further rescaling of the metric

$$g_{\mu\nu} \to k^{-2/(d-3)} g_{\mu\nu}\,, \tag{1.10}$$

brings us to the final form of the action:

$$S = \int d^{d-1}x\sqrt{|g|}\;\left[R + \tfrac{1}{2}\left(\partial\varphi\right)^2 - \tfrac{1}{4}e^{-a\varphi}F_{(2)}^2 + \tfrac{1}{2}\left(\mathcal{D}\phi\right)^2 - \tfrac{1}{2}m^2 e^{a\varphi}\right]\,, \tag{1.11}$$

where

$$k = e^{-\varphi/2a}\,, \qquad a = -\sqrt{\frac{2(d-2)}{(d-3)}}\,. \tag{1.12}$$

This action and the field strengths are invariant under the following massive gauge transformations:

$$\left\{\begin{array}{rcl}\delta\phi & = & m\chi\,,\\ \\ \delta A_{(1)\;\mu} & = & \partial_\mu\chi\,.\end{array}\right. \tag{1.13}$$

These transformations correspond in the d-dimensional theory to the z-independent reparametrizations of z:

$$\delta z = -\chi(x)\,. \tag{1.14}$$

This is the theory resulting from the standard recipe for generalized dimensional reduction [8].

There is another way of getting the same result in this toy model: We gauge the translation $\hat{\phi} \rightarrow \hat{\phi} + m$ and impose that the gauge field is non-vanishing and constant in the internal direction only (a Wilson line). Since the metric does not transform, it is sufficient to demonstrate this on the kinetic term for $\hat{\phi}$.

In order to gauge the translation invariance on $\hat{\phi}$ we introduce the gauge field by minimal coupling

$$\partial_{\hat{\mu}}\hat{\phi} \,\rightarrow\, \mathcal{D}_{\hat{\mu}}\hat{\phi} \,=\, \partial_{\hat{\mu}}\hat{\phi} \,+\, \hat{\mathcal{E}}_{\hat{\mu}}\,, \tag{1.15}$$

so that under a local transformation $\hat{\phi} \rightarrow \hat{\phi} + \Lambda(\hat{x})$ the gauge field transforms in an Abelian manner, i.e.

$$\hat{\mathcal{E}}'_{\hat{\mu}} \,=\, \hat{\mathcal{E}}_{\hat{\mu}} \,+\, \partial_{\hat{\mu}}\Lambda(\hat{x})\,. \tag{1.16}$$

Making then the *standard* KK Ansatz and imposing that $\hat{\mathcal{E}}_{\hat{\mu}}$ is non-vanishing and constant, with value m, in the compact direction only, one finds

$$\left\{\begin{array}{rcl}\mathcal{D}_a\phi & = & e_a{}^\mu\left(\partial_\mu\phi \,-\, mA_{(1)\;\mu}\right) \,\equiv\, e_a{}^\mu\mathcal{D}_\mu\phi\,,\\ \\ \mathcal{D}_z\phi & = & k^{-1}m\,,\end{array}\right. \tag{1.17}$$

leading to

$$\int d^dx\sqrt{|\hat{g}|}\,\tfrac{1}{2}\left(\partial\phi\right)^2 \,=\, \int d^{d-1}x\sqrt{|g|}\,k\left[\tfrac{1}{2}\left(\mathcal{D}\phi\right)^2 \,-\, \tfrac{1}{2}k^{-2}m^2\right]\,. \tag{1.18}$$

Comparing this result with Eq. (1.7), one sees that, at least in this toy-model, generalized Scherk-Schwarz reduction leads to the same result as the above algorithm.

We will also use this method in the context of type IIB supergravity and check that one gets the same results as well.

Observe that the field content looks the same as in the standard dimensional reduction: There is a vector and two scalars (apart from the metric). The symmetries and couplings are different, though. The massive gauge symmetry allows us to eliminate one scalar (the Stueckelberg field Ref. [22]) and give mass to the vector field. The number of degrees of freedom is exactly the same. So, what is it we have done? To shed some light on the meaning of this procedure we are going to perform the "standard" dimensional reduction of the action (1.1) but Poincaré-dualizing first the scalar into a $(d-2)$-form potential[9] $\hat{A}_{(d-2)\ \hat{\mu}_1\cdots\hat{\mu}_{(d-1)}}$:

$$\partial\hat{\phi} = {}^\star\hat{F}_{(d-1)}\,. \tag{1.19}$$

The dual action is

$$\tilde{\hat{S}} = \int d^d x\sqrt{|\hat{g}|}\ \left[\hat{R} + \tfrac{(-1)^{(d-2)}}{2\cdot(d-1)!}\hat{F}^2_{(d-1)}\right]\,. \tag{1.20}$$

Standard dimensional reduction with the same Vielbein Ansatz gives

$$\tilde{S} = \int d^{d-1}x\sqrt{|g|}\ k\ \left[R - \tfrac{1}{4}k^2 F^2_{(2)} + \tfrac{(-1)^{(d-2)}}{2\cdot(d-1)!}F^2_{(d-1)} + \tfrac{(-1)^{(d-3)}}{2\cdot(d-2)!}k^{-2}F^2_{(d-2)}\right]\,, \tag{1.21}$$

where

$$\left\{\begin{array}{rcl} F_{(d-1)} & = & (d-1)\partial A_{(d-2)} + (-1)^{(d-1)}A_{(1)}F_{(d-2)}\,, \\ & & \\ F_{(d-2)} & = & (d-2)\partial A_{(d-3)}\,, \end{array}\right. \tag{1.22}$$

are the field strengths of the $(d-2)$- and $(d-3)$-form potentials of the $(d-1)$-dimensional theory.

We can now dualize the potentials. A $(d-2)$-form potential in $(d-1)$ dimensions is dual to a constant that we call m. Adding the term

$$-\tfrac{1}{(d-1)!}\int d^{d-1}x\ m\epsilon\left[F_{(d-1)} + (-1)^d(d-1)A_{(1)}F_{(d-2)}\right]\,, \tag{1.23}$$

to the action (1.21), and eliminating $F_{(d-1)}$ using its equation of motion

$$m = k^\star F_{(d-1)}\,, \tag{1.24}$$

in the action we get

[9] When indices are not explicitly shown we assume all indices to be antisymmetrized with weight one. This is slightly different from differential form notation.

$$
\begin{aligned}
\tilde{S} \;=\; & \int d^{d-1}x\sqrt{|g|}\;\Big\{k\;\Big[R-\tfrac{1}{4}k^2F^2_{(2)}+\tfrac{(-1)^{(d-3)}}{2\cdot(d-2)!}k^{-2}F^2_{(d-2)}-\tfrac{1}{2}m^2k^{-2}\Big] \\
& +\tfrac{1}{(d-2)!}\tfrac{\epsilon}{\sqrt{|g|}}F_{(d-2)}\left[-mA_{(1)}\right]\Big\}\,.
\end{aligned}
\tag{1.25}
$$

Now we dualize into a scalar field the $(d-3)$-form potential: We add to the above action the term

$$
\tfrac{1}{(d-2)(d-2)!}\int d^{d-1}x\;\epsilon F_{(d-2)}\partial\phi\,, \tag{1.26}
$$

and eliminate $F_{(d-2)}$ by substituting in the action its equation of motion

$$
F_{(d-2)}=(-1)^{(d-2)}k\;{}^\star\mathcal{D}\phi\,, \tag{1.27}
$$

obtaining, perhaps surprisingly, Eq. (1.7).

What we have done is represented in figure 1.

The translation to brane language is obvious: Generalized dimensional reduction, which is essentially applied to scalars, is a way of keeping track of the dual $(d-3)$- and $(d-4)$-branes which should arise had we started with the dual of the scalar field.

Observe that in the generalized dimensional reduction Ansatz, Eq. (1.2), the scalar is not single-valued in the compact coordinate: $\hat{\phi}(z+1)=\hat{\phi}(z)+m$. The charge of the $(d-3)$-brane can be associated to the monodromy of $\hat{\phi}$ and to the $(d-1)$-dimensional vector mass:

$$
q\sim\int{}^\star\hat{F}_{(d-1)}\sim\int d\hat{\phi}\sim m\,. \tag{1.28}
$$

The implication of these results is obvious: The standard recipe for generalized dimensional reduction is just a way of performing a dimensional reduction taking into account all the possible fields (i.e. branes) that can arise in $(d-1)$ dimensions. In particular, the presence of $(d-3)$-branes is associated to the dependence on the internal coordinate and the charge of the background $(d-3)$-branes is proportional to the mass parameter. Generalized dimensional reduction should, from this point of view, be considered the standard full dimensional reduction, while the standard dimensional reduction is incomplete and there is an implicit truncation. The reason why this has not been realized before is that the missing fields only carry discrete degrees of freedom. The mass parameters are to be considered fields, although one can equally consider them as expectation values of those fields.

In the remainder of the paper we are going to perform a generalized dimensional reduction in the, more complex, context of type IIB supergravity. The underlying physics is, however, the same. The upshot is that what we are going to do is to perform the full dimensional reduction, without missing any fields as if we were able to Poincaré-dualize the type IIB scalars into 8-form potentials which is technically complicated.

Figure 1: This diagram represents two different ways of obtaining the same result: Generalized dimensional reduction and "dual" standard dimensional reduction.

Since we know that the type IIA and IIB string theories are T dual and we know that this implies the same for their low-energy (supergravity) theories, we expect T duality to keep working in the generalized dimensional reduction context. This poses several questions that we will also try to answer.

Thus, the contents and structure of the paper are as follows: In Section 2 we perform the generalized dimensional reduction of type IIB supergravity in an $SL(2,\mathbb{R})$-covariant way and obtain the massive 9-dimensional type II theory which is $SL(2,\mathbb{R})$-covariant. We analyze its global and local symmetries.

In Section 3 we obtain the same result using a new recipe for generalized dimensional reduction which involves the gauging of the global symmetry.

In Section 4 we study the M/type IIA side of the problem. First, we review the manifestly $SL(2,\mathbb{R})$-covariant compactification of 11-dimensional supergravity which gives the standard massless 9-dimensional type II theory and, then, in Section 4.1, we propose a massive generalization which, upon compactification on a T^2 gives precisely the massive 9-dimensional type II theory we obtained in the type IIB side of our problem. Figure 2 contains a schematic representation of the different dimensional reductions involved in this work.

In Section 5 we study 7-brane solutions and relate their monodromy properties with the mass matrix of the massive 9-dimensional type II theory. This allows us to give physical meaning to the mass parameters as 7-brane charges.

In Section 6 we study the duality relations between KK-type branes conjectured in this Introduction and depicted in Figure 4.

Section 7 contains our conclusions and some speculations and Figure 5 which synthesizes our present knowledge about the duality relations of the different extended M/string-theory solitons.

Finally in Appendix A and Appendix B we relate the fields of the massive 9-dimensional type II theory we have obtained with the 10-dimensional fields of the type IIA and B theories respectively. These relations imply the Buscher's T duality relations between the 10-dimensional fields themselves which are given in Appendix C.

Figure 2: Scheme of the different dimensional reductions with the names of the respective compact coordinates z, x, y.

2 The $Sl(2,\mathbb{R})$-Covariant Generalized Dimensional Reduction of Type IIB Supergravity: An S Duality Multiplet of $N=2, d=9$ Massive Supergravities

In this Section we perform the complete generalized dimensional reduction of type IIB supergravity in the direction parametrized by y (see Fig. 2) using the ideas of Ref. [8] as they were generalized in Ref. [11]. As we are going to explain, in the end we will obtain a three-parameter family (a triplet) of type II 9-dimensional supergravities connected by $SL(2,\mathbb{R})$ transformations (in the adjoint representation).

We are going to perform the generalized dimensional reduction in a manifestly $SL(2,\mathbb{R})$-covariant way. $SL(2,\mathbb{R})$ symmetry is manifest in the Einstein-frame. However, T duality, being a stringy symmetry, is better described in string frame. Thus we will spend some time relating the fields appearing in both frames. Since reducing an action is easier than reducing equations of motion, we are going to use the *non-self-dual* (NSD) action introduced in Ref. [23]. We study these two points in the following subsection and we perform the actual reduction in the next section.

2.1 An Overview of Type IIB Supergravity: The NSD Action and $Sl(2,R)$ Symmetry

It is well-known [24] that it is not possible to write a covariant action whose minimization gives the equations of motion 10-dimensional type IIB supergravity. The problematic equation of motion is the self-duality of the 5-form field strength. However, we can use it to find an alternative equation of motion just by replacing the 5-form field strength by its Hodge dual in the Bianchi identity. This alternative equation of motion has the conventional form of the equation of motion of a 4-form potential and it is possible to find an action from which to derive this and the other equations of motion *but not self-duality.* This NSD action, supplemented by the self-duality constraint gives all the equations of motion of the type IIB theory.

The (bosonic sector of the) string-frame NSD action is[10]

$$
\begin{aligned}
S_{\mathrm{NSD}} \;=\; & \int d^{10}\hat{x}\,\sqrt{|\hat{\jmath}|}\,\left\{e^{-2\hat{\varphi}}\left[\hat{R}(\hat{\jmath})-4\left(\partial\hat{\varphi}\right)^2+\tfrac{1}{2\cdot 5!}\hat{\mathcal{H}}^2\right]\right.\\
& +\tfrac{1}{2}\left(\hat{G}^{(0)}\right)^2+\tfrac{1}{2\cdot 3!}\left(\hat{G}^{(3)}\right)^2+\tfrac{1}{4\cdot 3!}\left(\hat{G}^{(5)}\right)^2 \\
& \left.-\tfrac{1}{192}\frac{1}{\sqrt{|\hat{\jmath}|}}\,\epsilon\,\partial\hat{C}^{(4)}\partial\hat{C}^{(2)}\hat{\mathcal{B}}\right\},
\end{aligned}
\tag{2.1}
$$

where $\{\hat{\jmath}_{\hat{\mu}\hat{\nu}},\hat{\mathcal{B}}_{\hat{\mu}\hat{\nu}},\hat{\varphi}\}$ are the NS-NS fields: The type IIB string metric, the type IIB NS-NS 2-form and the type IIB dilaton respectively.

$$
\hat{\mathcal{H}}_{\hat{\mu}\hat{\mu}\hat{\rho}}=3\partial_{[\hat{\mu}}\hat{\mathcal{B}}_{\hat{\nu}\hat{\rho}]}\,,\qquad\left(\hat{\mathcal{H}}=3\partial\hat{\mathcal{B}}\right),
\tag{2.2}
$$

is the NS-NS 2-form field strength. $\{\hat{C}^{(0)},\hat{C}^{(2)}{}_{\hat{\mu}\hat{\nu}},\hat{C}^{(4)}{}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\}$ are the RR potentials. Their field strengths and gauge transformations are

$$
\left\{
\begin{aligned}
\hat{G}^{(1)} &= \partial\hat{C}^{(0)}\,,\\
\hat{G}^{(3)} &= 3\left(\partial\hat{C}^{(2)}-\partial\hat{\mathcal{B}}\hat{C}^{(0)}\right),\\
\hat{G}^{(5)} &= 5\left(\partial\hat{C}^{(4)}-6\partial\hat{\mathcal{B}}\hat{C}^{(2)}\right).
\end{aligned}
\right.
\tag{2.3}
$$

and

[10]From now on we denote with hats and double hats 10- and 11-dimensional objects respectively. Our conventions are essentially those of Ref. [18] but we change the symbols denoting NS-NS fields in the type IIB theory to distinguish them from those of the type IIA. In particular we use the index-free notation of that reference: when indices are not explicitly shown, they are assumed to be completely antisymmetrized with weight one, and the definition of field strengths and gauge transformations are inspired by those of Refs. [25, 26].

$$
\left\{
\begin{array}{rcl}
\delta\hat{C}^{(0)} & = & 0\,, \\[1em]
\delta\hat{C}^{(2)} & = & 2\partial\hat{\Lambda}^{(1)}\,, \\[1em]
\delta\hat{C}^{(4)} & = & 4\partial\hat{\Lambda}^{(3)} + 6\hat{\mathcal{B}}\partial\hat{\Lambda}^{(1)}\,,
\end{array}
\right.
\tag{2.4}
$$

respectively.

The equations of motion derived from the above action have to be supplemented by the self-duality condition

$$
\hat{G}^{(5)} = +\,{}^{\star}\hat{G}^{(5)}\,. \tag{2.5}
$$

In the original version of the 10-dimensional, chiral $N=2$ supergravity [24] the theory has a classical $SU(1,1)$ global symmetry. The two scalars parametrize the coset $SU(1,1)/U(1)$, $U(1)$ being the maximal compact subgroup of $SU(1,1)$, and transform under a combination of a global $SU(1,1)$ transformation and a local $U(1)$ transformation which depends on the global $SU(1,1)$ transformation. They are combinations of the dilaton and the RR scalar. The group $SU(1,1)$ is isomorphic to $SL(2,\mathbb{R})$, the conjectured classical S duality symmetry group for the type IIB string theory [27]. A simple field redefinition [6] is enough to rewrite the action in terms of two real scalars parametrizing the coset $SL(2,\mathbb{R})/SO(2)$ which can now be identified with the dilaton $\hat{\varphi}$ and the RR scalar $\hat{C}^{(0)}$.

In order to make the S duality symmetry manifest, we first have to rescale the metric as to go to the Einstein frame:

$$
\hat{\jmath}_{E\ \hat{\mu}\hat{\nu}} = e^{-\hat{\varphi}/2}\hat{\jmath}_{\hat{\mu}\hat{\nu}}\,. \tag{2.6}
$$

We now have to make some further field redefinitions. For instance, while the NS-NS and RR 2-forms we are using form an $SL(2,\mathbb{R})$ doublet, their field strengths do not. Furthermore, our self-dual RR 4-form potential $\hat{C}^{(4)}$ is not $SL(2,\mathbb{R})$-invariant. Thus, for the purpose of exhibiting the $SL(2,\mathbb{R})$ symmetry it is convenient to perform the following field redefinitions[11]:

$$
\left\{
\begin{array}{rcl}
\hat{\vec{\mathcal{B}}} & = & \begin{pmatrix} \hat{C}^{(2)} \\ \hat{\mathcal{B}} \end{pmatrix}\,, \\[1.5em]
\hat{D} & = & \hat{C}^{(4)} - 3\hat{\mathcal{B}}\hat{C}^{(2)}\,,
\end{array}
\right.
\tag{2.7}
$$

These new fields undergo the following gauge transformations:

[11] Our conventions are such that all fields are either invariant or transform *covariantly* as opposed to *contravariantly*.

$$
\begin{cases}
\delta\hat{\vec{\mathcal{B}}} &= 2\hat{\vec{\Sigma}}\,, \\[2ex]
\delta\hat{D} &= 4\partial\hat{\Delta} + 2\hat{\vec{\Sigma}}^{\,T}\eta\,\hat{\vec{\mathcal{H}}}\,,
\end{cases}
\tag{2.8}
$$

and have field strengths

$$
\begin{cases}
\hat{\vec{\mathcal{H}}} &= 3\partial\hat{\vec{\mathcal{B}}}\,, \\[2ex]
\hat{F} &= \hat{G}^{(5)} = +\,{}^{\star}\hat{F} \\[2ex]
 &= 5\left(\partial\hat{D} - \hat{\vec{\mathcal{B}}}^{\,T}\eta\,\hat{\vec{\mathcal{H}}}\right)\,,
\end{cases}
\tag{2.9}
$$

where η is the 2×2 matrix

$$
\eta = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\eta^{-1} = -\eta^T\,,
\tag{2.10}
$$

Given the isomorphism $SL(2,\mathbb{R}) \sim Sp(2,\mathbb{R})$, it can be identified with an invariant metric:

$$
\Lambda\eta\Lambda^T = \eta\,, \;\Rightarrow\; \eta\Lambda\eta^T = (\Lambda^{-1})^T\,, \qquad \Lambda \in SL(2,\mathbb{R})\,.
\tag{2.11}
$$

Finally, it is convenient to define the 2×2 matrix $\hat{\mathcal{M}}_{ij}$

$$
\hat{\mathcal{M}} = e^{\hat{\varphi}}\begin{pmatrix} |\hat{\lambda}|^2 & \hat{C}^{(0)} \\[2ex] \hat{C}^{(0)} & 1 \end{pmatrix}\,, \qquad
\hat{\mathcal{M}}^{-1} = e^{\hat{\varphi}}\begin{pmatrix} 1 & -\hat{C}^{(0)} \\[2ex] -\hat{C}^{(0)} & |\hat{\lambda}|^2 \end{pmatrix}\,,
\tag{2.12}
$$

where $\hat{\lambda}$ is the complex scalar

$$
\hat{\lambda} = \hat{C}^{(0)} + ie^{-\hat{\varphi}}\,.
\tag{2.13}
$$

Observe that $\hat{\mathcal{M}}$ is a symmetric $SL(2,\mathbb{R})$ matrix and therefore, as a consequence of Eq. (2.11) it has the property

$$
\hat{\mathcal{M}}^{-1} = \eta\hat{\mathcal{M}}\eta^T\,,
\tag{2.14}
$$

To see that $\hat{\lambda}$ parametrizes the $SL(2,\mathbb{R})/SO(2)$ coset, it is convenient to consider how one arrives at $\hat{\mathcal{M}}$. First one considers the non-symmetric $SL(2,\mathbb{R})$ matrix $\hat{V}$

$$
\hat{V} = \begin{pmatrix} e^{-\hat{\varphi}/2} & e^{\hat{\varphi}/2}\hat{C}^{(0)} \\[2ex] 0 & e^{\hat{\varphi}/2} \end{pmatrix}\,.
\tag{2.15}
$$

This $SL(2,\mathbb{R})$ matrix is generated by only two of the three $SL(2,\mathbb{R})$ generators and it should cover the $SL(2,\mathbb{R})/SO(2)$ coset. The choice for the form of $\hat{V}$ can be understood as a choice of gauge or as a choice of coset representatives. However, an arbitrary $SL(2,\mathbb{R})$ transformation Λ will transform $\hat{V}$ into a non-upper-triangular matrix $\Lambda\hat{V}$ (which is not a coset representative). A further Λ-dependent $SO(2)$-transformation h will, by using the definition of a coset, take us to another coset representative $\hat{V}' = \Lambda\hat{V}h$. The transformation h will be *local* but not arbitrary. It can be thought of as a compensating gauge transformation. The condition that $\hat{V}'$ is upper-triangular fully determines $h(\Lambda,\hat{V})$ and the transformations of $\hat{C}^{(0)}$ and $\hat{\varphi}$:

$$
\begin{aligned}
\hat{V}' &= \begin{pmatrix} e^{-\hat{\varphi}'/2} & e^{\hat{\varphi}'/2}\hat{C}^{(0)\,\prime} \\ 0 & e^{\hat{\varphi}'/2} \end{pmatrix} = \Lambda\hat{V}h = \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{-\hat{\varphi}/2} & e^{\hat{\varphi}/2}\hat{C}^{(0)} \\ 0 & e^{\hat{\varphi}/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},
\end{aligned}
\tag{2.16}
$$

where $ad-bc=1$. The result is that the parameter θ of the compensating transformation h is given by

$$
\tan\theta = \frac{c}{e^{\hat{\varphi}}\left(c\hat{C}^{(0)}+d\right)}\,,
\tag{2.17}
$$

and the transformation of the scalars can be written in the compact form

$$
\hat{\lambda}' = \frac{a\hat{\lambda}+b}{c\hat{\lambda}+d}\,.
\tag{2.18}
$$

The symmetric matrix $\hat{\mathcal{M}}$ is now $\hat{\mathcal{M}} = \hat{V}\hat{V}^T$ and transforms under $\Lambda \in SL(2,\mathbb{R})$ according to

$$
\hat{\mathcal{M}}' = \Lambda\hat{\mathcal{M}}\Lambda^T\,,
\tag{2.19}
$$

which is completely equivalent to the above transformation of $\hat{\lambda}$. Observe that it is not necessary to worry about the h-transformations anymore.

It is also worth stressing that the only $SL(2,\mathbb{R})$ transformations that leave invariant $\hat{\lambda}$ or, equivalently, $\hat{\mathcal{M}}$ or $\hat{V}$ are $\pm\mathbb{I}_{2\times 2}$. This is an important point: $SO(2)$ is sometimes referred to as the "stability subgroup". Had we defined the coset by the equivalence relation $\hat{V} \sim h\hat{V}\,,\ h\in SO(2)$, then, by definition, $\hat{V}$ would have been invariant under any $\Lambda \in SO(2)$. Then, $SO(2)$ would have been the subgroup of $SL(2,\mathbb{R})$ leaving invariant the coset scalars. This is, however, *not* the way in which this coset is constructed and (as it can be explicitly checked) there is no stability subgroup of $SL(2,\mathbb{R})$ in that sense apart from this almost trivial $\mathbb{Z}_2$.

Under this Λ, the doublet of 2-forms transforms

$$
\hat{\vec{\mathcal{B}}}' = \Lambda\hat{\vec{\mathcal{B}}}\,,
\tag{2.20}
$$

and the 4-form $\hat{D}$ and the Einstein metric are inert.

Now, it is a simple exercise to rewrite the NSD type IIB action in the following manifestly S duality invariant form

$$
\begin{aligned}
\hat{S}_{\rm NSD} \;\; = \;\; & \tfrac{1}{16\pi G_N^{(10)}} \int d^{10}\hat{x}\;\sqrt{|\hat{\jmath}_E|}\left\{\hat{R}(\hat{\jmath}_E) + \tfrac{1}{4}{\rm Tr}\left(\partial\hat{\mathcal{M}}\hat{\mathcal{M}}^{-1}\right)^2\right. \\
& \left. + \tfrac{1}{2\cdot 3!}\hat{\vec{\mathcal{H}}}^{\;T}\hat{\mathcal{M}}^{-1}\hat{\vec{\mathcal{H}}} + \tfrac{1}{4\cdot 3!}\hat{F}^2 - \tfrac{1}{2^7\cdot 3^3}\tfrac{1}{\sqrt{|\hat{\jmath}_E|}}\,\epsilon\,\hat{D}\,\hat{\vec{\mathcal{H}}}^{\;T}\eta\,\hat{\vec{\mathcal{H}}}\right\}\,,
\end{aligned}
\tag{2.21}
$$

It is easy to find how the fields $\hat{\mathcal{H}}, \hat{G}^{(3)}, \hat{C}^{(4)}$ in the action Eq. (2.1) transform under $SL(2,\mathbb{R})$:

$$
\left\{
\begin{array}{rcl}
\hat{\mathcal{H}}' & = & \left(d + c\hat{C}^{(0)}\right)\hat{\mathcal{H}} + c\hat{G}^{(3)}\,, \\[2ex]
\hat{G}^{(3)\;\prime} & = & \frac{1}{|c\hat{\lambda}+d|^2}\left[\left(d + c\hat{C}^{(0)}\right)\hat{G}^{(3)} - ce^{-2\hat{\varphi}}\hat{\mathcal{H}}\right]\,, \\[2ex]
\hat{C}^{(4)\;\prime} & = & \hat{C}^{(4)} - 3\left(\begin{array}{cc}\hat{C}^{(2)} & \hat{\mathcal{B}}\end{array}\right)\left(\begin{array}{cc} ac & bc \\ bc & db\end{array}\right)\left(\begin{array}{c}\hat{C}^{(2)} \\ \hat{\mathcal{B}}\end{array}\right)\,.
\end{array}
\right.
\tag{2.22}
$$

$\hat{\lambda}$ transforms as above and we stress that the string metric does transform under $SL(2,\mathbb{R})$:

$$
\hat{\jmath}' = |c\hat{\lambda} + d|\hat{\jmath}\,. \tag{2.23}
$$

2.2 Generalized Dimensional Reduction

Now that we have set up the action we want to reduce, we can proceed. First, we will explain the generalized KK Ansatz. In this point we will follow the recipe of Ref. [11] adapted to our conventions. Then we will reduce the action and the self-duality constraint and finally we will eliminate the constraint, obtaining the action of the 9-dimensional theory.

The fields of the Einstein-frame 9-dimensional theory are the same as in the massless case:

$$
\{g_{E\;\mu\nu}, A_{(3)\;\mu\nu\rho}, \vec{A}_{(2)\;\mu\nu}, \vec{A}_{(1)\;\mu}, A_{(1)\;\mu}, K, \mathcal{M}\}\,, \tag{2.24}
$$

and only the couplings and symmetries will be different.

2.2.1 The Kaluza-Klein Ansatz

As usual in dimensional reductions, we assume the existence of a Killing vector $\hat{s}^{\hat{\mu}}\partial_{\hat{\mu}} = \partial_{\underline{y}}$ associated to the coordinate y. We choose adapted coordinates $\hat{x}^{\hat{\mu}} = (x^\mu, y)$ so that the metric does not depend on y. We normalize the coordinate y such that it takes values in the interval $[0,1]$ and so $y \sim y+1$. Our Ansatz for the Einstein-frame Zehnbeins is then that of Eq. (1.6) adapted to ten dimensions and with the scalar k, the length of the (spacelike) Killing vector, relabeled

$$|\hat{s}^{\hat{\mu}}\hat{s}_{\hat{\mu}}|^{1/2} = K^{-3/4}\,, \tag{2.25}$$

for convenience.

Now, instead of assuming that all the other fields in our theory have vanishing Lie derivatives with respect to $\hat{s}^{\hat{\mu}}$, we assume that the remaining fields depend on y but in a very specific way: All the y-dependence is introduced by a local $SL(2,\mathbb{R})$ transformation with parameters linear in y, $\Lambda(y)$:

$$\left\{\begin{array}{rcl} \hat{\mathcal{M}}(\hat{x}) & \equiv & \Lambda(y)\hat{\mathcal{M}}^{\rm b}(x)\Lambda^T(y)\,, \\ \hat{\vec{\mathcal{B}}}(\hat{x}) & \equiv & \Lambda(y)\vec{\mathcal{B}}^{\rm b}(x)\,, \\ \hat{D}(\hat{x}) & = & \hat{D}^{\rm b}(x)\,, \end{array}\right. \tag{2.26}$$

where we have denoted by a superscript ^{b} the *bare* y-independent fields.

Obviously, the Ansatz for $\hat{\mathcal{M}}$ is equivalent, in terms of $\hat{\lambda}(\hat{x})$ to

$$\hat{\lambda}(\hat{x}) = \frac{a(y)\hat{\lambda}^{\rm b}(x) + b(y)}{c(y)\hat{\lambda}^{\rm b}(x) + d(y)}\,. \tag{2.27}$$

In this scheme $\hat{D}$ cannot depend on y because it is inert under $SL(2,\mathbb{R})$, but it is worth stressing that the string-frame metric does depend on y. The *bare* fields are y-independent and will become the 9-dimensional fields. On the other hand, they transform under $SL(2,\mathbb{R}$ as the real fields do.

The meaning of this kind of Ansatz is the following: We are constructing a non-trivial line bundle over the circle parametrized by y with fiber $\hat{\lambda}$ (or, equivalently $\hat{\mathcal{M}}$) and structure group $SL(2,\mathbb{R})$ (we will later study the restriction to $SL(2,\mathbb{Z})$). Going once around the circle we go back to the same $\hat{\mathcal{M}}$ up to a global $SL(2,\mathbb{R})$ transformation that we can describe by an $SL(2,\mathbb{R})$ monodromy matrix M. The explicit form of M depends on the explicit form of $\Lambda(y)$.

Let us now describe more precisely the form of $\Lambda(y)$. If

$$T_1 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad T_2 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad T_3 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{2.28}$$

are the generators of $SL(2,\mathbb{R})$, then the most general $SL(2,\mathbb{R})$ transformation with local parameters linear in y can be written in the form

$$\Lambda(y) = \exp\left\{\tfrac{1}{2}ym^iT_i\right\}. \tag{2.29}$$

The three real parameters m^i fully determine $\Lambda(y)$ and therefore the particular compactification. These parameters are going to become masses in the lower-dimensional theory. We define the *mass matrix* m

$$m \equiv \left(\partial_{\underline{y}}\Lambda\right)\Lambda^{-1} = \tfrac{1}{2}m^i T_i = \tfrac{1}{2}\begin{pmatrix} m^1 & m^2+m^3 \\ \\ m^2-m^3 & -m_1 \end{pmatrix} . \tag{2.30}$$

This matrix belongs to the Lie algebra $sl(2,\mathbb{R})$ and therefore it transforms in the (irreducible) adjoint representation:

$$m' = \Lambda m \Lambda^{-1} , \tag{2.31}$$

and thus the three m^i transform as a triplet (a vector of $SO(2,1) \sim SL(2,\mathbb{R})$). The expression

$$\alpha^2 = \mathrm{Tr}\ (m^2) = \tfrac{1}{4}m^i m^j h_{ij} , \qquad h_{ij} = \mathrm{diag}(++-) , \tag{2.32}$$

where h_{ij} is the Killing metric, is thus $SL(2,\mathbb{R})$-invariant. Furthermore, the mass matrix satisfies

$$\eta m \eta^{-1} = -m^T . \tag{2.33}$$

Observe that the parameters m^1, m^2 are associated to non-compact generators of $SL(2,\mathbb{R})$, while m^3 is associated to the maximal compact subgroup of $SL(2,\mathbb{R})$ ($SO(2)$). Thus, we are bound to get mass terms with the wrong sign (for instance in terms like Eq. (2.32)) but we must keep the three mass parameters in order to have full $SL(2,\mathbb{R})$-covariance and the most general 9-dimensional massive type II supergravity.

Our Ansatz generalizes that of Ref. [11], which only had two independent parameters: $m^1, m^2 = m^3$. The authors argued that generalized dimensional reduction using $SL(2,\mathbb{R})$ y-dependent transformations in the stability subgroup $SO(2)$ (i.e. those generated by T_3 and associated to m^3 in our conventions) would have no effect. As we discussed in the previous Section, there is no stability subgroup for the coset scalars. Furthermore, since the three mass parameters we just defined transform irreducibly, the three of them are required to obtain $SL(2,\mathbb{R})$-covariant families of theories. Finally, the $SL(2,\mathbb{R})$ transformation $S=\eta$ is inside the excluded $SO(2)$ and this is one of the generators of the quantum S duality group $SL(2,\mathbb{Z})$.

$\Lambda(y)$ will only manifest itself through the mass matrix in the lower-dimensional theory. However, in order to reconstruct the 10-dimensional fields we need to know it explicitly. The explicit form of $\Lambda(y)$ reads

$$\Lambda(y) = \begin{pmatrix} \cosh\alpha y + \frac{m^1}{2\alpha}\sinh\alpha y & \frac{m^2+m^3}{2\alpha}\sinh\alpha y \\ \\ \frac{m^2-m^3}{2\alpha}\sinh\alpha y & \cosh\alpha y - \frac{m^1}{2\alpha}\sinh\alpha y \end{pmatrix} , \tag{2.34}$$

where α was defined in Eq. (2.32).

It is easy to see from our definition of $\Lambda(y)$ that the monodromy matrix will be

$$M(m^i) = \exp\{\tfrac{1}{2}m^i T_i\} = \Lambda(y=1)\,, \tag{2.35}$$

and

$$\left\{\begin{array}{rcl} \hat{\mathcal{M}}(x,y+1) & = & M\hat{\mathcal{M}}(x,y)M^T\,, \\ \\ \hat{\vec{\mathcal{B}}}(x,y+1) & = & M\vec{\mathcal{B}}(x,y)\,. \end{array}\right. \tag{2.36}$$

Quantum-mechanically, the monodromy matrices can only be $SL(2,\mathbb{Z})$ matrices. It is convenient to describe the most general $SL(2,\mathbb{Z})$ monodromy matrix by for integers $n^i, n\,,\ i=1,2,3$ subject to the constraint

$$n^i n_i = n^2 - 1\,. \tag{2.37}$$

Given that this constraint is satisfied, then we simply make the identifications

$$\alpha = \cosh^{-1} n\,, \qquad m^i = \frac{2\alpha}{\sqrt{n^2-1}} n^i\,, \tag{2.38}$$

and write the monodromy matrix as follows:

$$M = \begin{pmatrix} n+n^1 & n^2+n^3 \\ \\ n^2-n^3 & n-n^1 \end{pmatrix}\,. \tag{2.39}$$

Thus, in our conventions, the mass parameters m^i will be naturally quantized in terms of the three integers n^i which also transform in the “adjoint” of $SL(2,\mathbb{Z})$. n is $SL(2,\mathbb{Z})$-invariant.

In Section 5 we will relate the integers n^i to the charges of 7-branes.

We can now perform the dimensional reduction.

2.2.2 Dimensional Reduction

Using the standard techniques [7] we get with the just-described Ansatz the NSD 9-dimensional action

$$
\begin{aligned}
S_{\rm NSD} \;\; = & \\
\int d^9x\sqrt{|g|}\;\Big\{ & K^{-3/4}\;\Big[\;R(g)\;+\tfrac{1}{4}{\rm Tr}\left(D\mathcal{M}\mathcal{M}^{-1}\right)^2-\tfrac{1}{4}K^{-3/2}F_{(2)}^2 \\
& -\tfrac{1}{4}K^{3/2}\vec{F}_{(2)}^{\,T}\mathcal{M}^{-1}\vec{F}_{(2)}+\tfrac{1}{2\cdot 3!}\vec{F}_{(3)}^{\,T}\mathcal{M}^{-1}\vec{F}_{(3)}-\tfrac{1}{4\cdot 4!}K^{3/2}F_{(4)}^2 \\
& +\tfrac{1}{4\cdot 5!}F_{(5)}^2-K^{3/2}\mathcal{V}\left(\mathcal{M}\right)\Big] \\
& +\tfrac{1}{2^7\cdot 3^2\cdot 5}\tfrac{1}{\sqrt{|g|}}\;\epsilon\left\{\left(F_{(5)}-5A_{(1)}F_{(4)}\right)\times\right. \\
& \times\left[2\left(\vec{F}_{(3)}-3A_{(1)}\vec{F}_{(2)}\right)^T\eta\vec{A}_{(1)}+3\vec{F}_{(2)}^{\,T}\eta\vec{A}_{(2)}\right] \\
& \left.-5F_{(4)}\left(\vec{F}_{(3)}-3A_{(1)}\vec{F}_{(2)}\right)^T\eta\vec{A}_{(2)}\right\}\;,
\end{aligned}
\tag{2.40}
$$

and the 9-dimensional duality constraint

$$
F_{(5)}=-K^{3/4}\;{}^{\star}F_{(4)}\;, \tag{2.41}
$$

where the field strengths are defined as follows:

$$
\left\{
\begin{aligned}
\mathcal{D}\mathcal{M} &= \partial\mathcal{M}-\left(m\mathcal{M}+\mathcal{M}m^T\right)A_{(1)}\;,\\
F_{(2)} &= 2\partial A_{(1)}\;,\\
\vec{F}_{(2)} &= 2\partial\vec{A}_{(1)}-m\vec{A}_{(2)}\;,\\
\vec{F}_{(3)} &= 3\partial\vec{A}_{(2)}+3A_{(1)}\vec{F}_{(2)}\;,\\
F_{(4)} &= 4\partial A_{(3)}-3\vec{A}_{(2)}^{\,T}\;\eta\;\vec{F}_{(2)}+2\vec{A}_{(1)}^{\,T}\;\eta\;\vec{F}_{(3)}+6A_{(1)}\vec{A}_{(1)}^{\,T}\;\eta\;\vec{F}_{(2)}\;,\\
F_{(5)} &= 5\partial A_{(4)}-5\vec{A}_{(2)}^{\,T}\;\eta\;\vec{F}_{(3)}+15A_{(1)}\vec{A}_{(2)}{}^T\eta\vec{F}_{(2)}+5A_{(1)}F_{(4)}\;,
\end{aligned}
\right.
\tag{2.42}
$$

and

$$
\mathcal{V}\left(\mathcal{M}\right)=\tfrac{1}{2}{\rm Tr}\left(m^2+m\mathcal{M}m^T\mathcal{M}^{-1}\right)\;, \tag{2.43}
$$

is the scalar potential.

The 10- and 9-dimensional fields are related as follows:

$$
\begin{aligned}
\hat{\mathcal{M}}^{\mathrm{b}} &= \mathcal{M}\,, & \hat{D}_{\mu_1\mu_2\mu_3\underline{y}} &= -A_{(3)\ \mu_1\mu_2\mu_3}\,,\\
\vec{\mathcal{B}}^{\mathrm{b}}{}_{\mu\underline{y}} &= -\vec{A}_{(1)\ \mu}\,, & \hat{D}_{\mu_1\cdots\mu_4} &= A_{(4)\ \mu_1\cdots\mu_4}\,.\\
\vec{\mathcal{B}}^{\mathrm{b}}{}_{\mu\nu} &= \vec{A}_{(2)\ \mu\nu}\,,
\end{aligned}
\tag{2.44}
$$

2.2.3 Elimination of the Self-Duality Constraint and Rescaling of the Metric

In order to eliminate the self-duality constraint Eq. (2.41) we first Poincaré-dualize the NSD action with respect to the 4-form potential. First, we add the Lagrange multiplier term

$$
\begin{aligned}
&\tfrac{1}{2^5\cdot 3^2}\int d^9x\ \epsilon\partial\tilde{A}_{(3)}\partial A_{(4)} \quad = \\
&\tfrac{1}{2^5\cdot 3^2}\int d^9x\ \epsilon\partial\tilde{A}_{(3)}\left[F_{(5)} + 5\vec{A}_{(2)}^{\,T}\ \eta\ \vec{F}_{(3)} - 15A_{(1)}\vec{A}_{(2)}{}^{T}\eta\vec{F}_{(2)} - 5A_{(1)}F_{(4)}\right]\,,
\end{aligned}
\tag{2.45}
$$

to the NSD action (2.40). The equation of motion of the Lagrange multiplier field $\tilde{A}_{(3)}$ enforces the Bianchi identity of $F_{(5)}$ and we can consider the new action as a functional of $F_{(5)}$ instead of $A_{(4)}$ which does not occur explicitly. The equation of motion for $F_{(5)}$ is nothing but

$$
F_{(5)} = -K^{3/4}\ {}^{\star}\tilde{F}_{(4)}\,,
\tag{2.46}
$$

where $\tilde{F}_{(4)}$ is like $F_{(4)}$ but with $A_{(4)}$ replaced by $\tilde{A}_{(4)}$. This equation is purely algebraic and we can use it to eliminate $F_{(5)}$ in the NSD action (2.40) plus the Lagrange multiplier term. The result is an action the depends both on $A_{(4)}$ and $\tilde{A}_{(4)}$. Now, we simply observe that the equation of motion for $F_{(5)}$ has the same form as the self-duality constraint Eq. (2.41) and therefore, eliminating the self-duality constraint amounts to the simple identification

$$
F_{(4)} = \tilde{F}_{(4)}\,.
\tag{2.47}
$$

The result of these manipulations plus a Weyl rescaling to go to the Einstein frame (the metric g is neither the string metric nor Einstein's)

$$
g_{\mu\nu} = K^{3/14} g_{E\ \mu\nu}\,.
\tag{2.48}
$$

is the action of the type II massive supergravity:

$$
\begin{aligned}
S \;=\; & \int d^9x\,\sqrt{|g_E|}\,\Big\{R_E + \tfrac{9}{14}\left(\partial\log K\right)^2 + \tfrac{1}{4}\mathrm{Tr}\left(\mathcal{D}\mathcal{M}\mathcal{M}^{-1}\right)^2 - \tfrac{1}{4}K^{-12/7}F^2_{(2)} \\
& -\tfrac{1}{4}K^{\frac{9}{7}}\vec{F}^{\,T}_{(2)}\mathcal{M}^{-1}\vec{F}_{(2)} + \tfrac{1}{2\cdot 3!}K^{-3/7}\vec{F}^{\,T}_{(3)}\mathcal{M}^{-1}\vec{F}_{(3)} - \tfrac{1}{2\cdot 4!}K^{6/7}F^2_{(4)} - K^{12/7}\mathcal{V}\left(\mathcal{M}\right) \\
& -\tfrac{1}{2^7\cdot 3^2}\tfrac{1}{\sqrt{|g_E|}}\,\epsilon\,\Big\{16(\partial A_{(3)})^2A_{(1)} \\
& +24\partial A_{(3)}\left[\partial\vec{A}^{\,T}_{(2)}\eta\vec{A}_{(2)} - \left(4\vec{A}^{\,T}_{(2)}\eta\partial\vec{A}_{(1)} + 2\vec{A}^{\,T}_{(1)}\eta\partial\vec{A}_{(2)} - \vec{A}^{\,T}_{(2)}\eta m\vec{A}_{(2)}\right)A_{(1)}\right] \\
& -36\left(\vec{A}^{\,T}_{(2)}\eta\partial\vec{A}_{(1)} + \vec{A}^{\,T}_{(1)}\eta\partial\vec{A}_{(2)}\right)\partial\vec{A}^{\,T}_{(2)}\eta\vec{A}_{(2)} \\
& -36\left(\vec{A}^{\,T}_{(2)}\eta\partial\vec{A}_{(1)} - \vec{A}^{\,T}_{(1)}\eta\partial\vec{A}_{(2)}\right)^2A_{(1)} \\
& +9\vec{A}^{\,T}_{(2)}\eta m\vec{A}_{(2)}\left[\partial\vec{A}^{\,T}_{(2)}\eta\vec{A}_{(2)} - 4\left(\vec{A}^{\,T}_{(2)}\eta\partial\vec{A}_{(1)} - \vec{A}^{\,T}_{(1)}\eta\partial\vec{A}_{(2)}\right)A_{(1)}\right. \\
& \left.+(\,\vec{A}^{\,T}_{(2)}\eta m\vec{A}_{(2)}\,)A_{(1)}\right]\Big\}\Big\}\,.
\end{aligned}
\tag{2.49}
$$

whose topological term, in order to facilitate comparison with the results of Section 4, was rewritten in terms of potentials only (no field strengths) by integrating several times by parts and using algebraic properties like

$$
\left(\vec{A}^{\,T}_{(1)}\eta\partial\vec{A}_{(2)}\right)\left(\partial\vec{A}^{\,T}_{(2)}\eta\vec{A}_{(2)}\right) = -\tfrac{1}{2}\left(\vec{A}^{\,T}_{(1)}\eta\vec{A}_{(2)}\right)\left(\partial\vec{A}^{\,T}_{(2)}\eta\partial\vec{A}_{(2)}\right)\,. \tag{2.50}
$$

2.2.4 Gauge and Global Symmetries of the 9-Dimensional Theory

The local symmetries of the 9-dimensional theory (2.49) have three different origins: The gauge transformations of the 2-form fields:

$$
\delta\hat{\vec{\mathcal{B}}} = 2\partial\hat{\vec{\Sigma}}\,, \tag{2.51}
$$

the gauge transformations of the 4-form

$$
\delta\hat{D} = 4\partial\hat{\Delta} - \tfrac{2}{5}\hat{\vec{\Sigma}}^T\eta\hat{\vec{\mathcal{H}}}\,, \tag{2.52}
$$

and the y-independent reparametrizations of the compact coordinate y

$$
\delta\hat{x}^{\hat{\mu}} = \delta^{\hat{\mu}\underline{y}}\chi(x)\,. \tag{2.53}
$$

The dependence of the 10-dimensional fields on y, inexistent in standard dimensional reduction, induces new terms (the transport terms) in the χ-transformations.

The 9-dimensional fields have the following *infinitesimal* χ gauge transformations and finite $\Sigma_{(0)}, \vec{\Sigma}_{(0)}, \vec{\Sigma}_{(1)}, \Sigma_{(3)}$ gauge transformations:

$$
\left\{
\begin{array}{rcl}
\delta\mathcal{M} & = & \chi\left(m\mathcal{M} + \mathcal{M}m^T\right)\,, \\
\delta A_{(1)} & = & \partial\chi\,, \\
\delta\vec{A}_{(1)} & = & \partial\vec{\Sigma}_{(0)} + m\vec{\Sigma}_{(1)} + \chi m\vec{A}_{(1)}\,, \\
\delta\vec{A}_{(2)} & = & 2\partial\vec{\Sigma}_{(1)} + 2\partial\chi\vec{A}_{(1)} + \chi m\vec{A}_{(2)}\,, \\
\delta A_{(3)} & = & 3\partial\Sigma_{(2)} + \frac{3}{2}\vec{\Sigma}_{(1)}{}^T\eta\vec{F}_{(2)} - \frac{3}{2}\vec{\Sigma}_{(0)}{}^T\eta\partial\vec{A}_{(2)}\,, \\
\delta A_{(4)} & = & 4\partial\Sigma_{(3)} + 6\vec{\Sigma}_{(1)}\eta\partial\vec{A}_{(2)} + 4\partial\chi A_{(3)}\,.
\end{array}
\right.
\tag{2.54}
$$

The χ-transformations can be exponentiated:

$$
\left\{
\begin{array}{rcl}
V' & = & e^{\chi m}V\,, \\
\mathcal{M}' & = & e^{\chi m}\mathcal{M}\; e^{\chi m^T}\,, \\
A'_{(1)} & = & A_{(1)} + \partial\chi\,, \\
\vec{A}'_{(1)} & = & e^{\chi m}\vec{A}_{(1)}\,,
\end{array}
\right.
\qquad
\left\{
\begin{array}{rcl}
\vec{A}'_{(2)} & = & e^{\chi m}\left(\vec{A}_{(2)} + 2\partial\chi\vec{A}_{(1)}\right)\,, \\
A'_{(3)} & = & A_{(3)}\,, \\
A'_{(4)} & = & 4\partial\chi A_{(3)}\,.
\end{array}
\right.
\tag{2.55}
$$

Under the χ-transformations, the field strengths transform covariantly instead of being invariant:

$$
\left\{
\begin{array}{rcl}
(D\mathcal{M})' & = & e^{\chi m}D\mathcal{M}\; e^{\chi m^T}\,, \\
\vec{F}'_{(2,3)} & = & e^{\chi m}\vec{F}_{(2,3)}\,, \\
F'_{(4,5)} & = & F_{(4,5)}\,.
\end{array}
\right.
\tag{2.56}
$$

We could easily define field strengths invariant under χ-transformations: For instance

$$
\tilde{\vec{F}}_{(2,3)} = V^{-1}\vec{F}_{(2,3)}\,,
\tag{2.57}
$$

as was done in Ref. [8], but we will choose not to do so.

It is trivial to check the invariance of the action (2.49) under the above gauge transformations.

The action Eq. (2.49) enjoys some global invariances as well, namely rescalings of K and $SL(2,\mathbb{R})$ transformations. The latter are the most interesting. Their action on the fields $\mathcal{M}, \vec{A}_{(1)\;\mu}, \vec{A}_{(2)\;\mu\nu}$ is

$$\mathcal{M}' = \Lambda\mathcal{M}\Lambda^T\,, \qquad \vec{A}'_{(1,2)} = \Lambda\vec{A}_{(1,2)}\,. \tag{2.58}$$

As was said before, the mass matrix belongs to the Lie algebra $sl(2,\mathbb{R})$ and transforms in the adjoint representation:

$$m' = \Lambda m \Lambda^{-1}\,, \tag{2.59}$$

and thus the three m^i transform as a triplet (a vector of $SO(2,1) \sim SL(2,\mathbb{R})$).

Finally, the theory is also invariant under constant rescalings of the fields:

$$\begin{array}{rclcrcl} K & \to & e^{14\alpha}K\,, & \qquad & m & \to & e^{-12\alpha}m\,, \\[2ex] A_{(1)} & \to & e^{12\alpha}A_{(1)}\,, & & \vec{A}_{(1)} & \to & e^{-9\alpha}\vec{A}_{(1)}\,, \\[2ex] A_{(3)} & \to & e^{-6\alpha}A_{(3)}\,, & & \vec{A}_{(2)} & \to & e^{3\alpha}\vec{A}_{(2)}\,. \end{array} \tag{2.60}$$

3 An Alternative Recipe for Generalized Dimensional Reduction: Gauging of Global Symmetries

In this Section we will apply an alternative recipe for generalized dimensional reduction to type IIB supergravity. The general idea is that gauging the global symmetry and imposing that the gauge field takes non-vanishing and constant values in the internal direction only, is equivalent to applying generalized Scherk-Schwarz reduction. In order to demonstrate this, the algorithm will be applied to the NSD IIB action, albeit written in terms of forms. The conventions for forms are the ones used in Ref. [28] and in particular we need

$$\int F_{(p)}{}^{\star}F_{(p)} = \int d^dx\sqrt{|g|}\;\tfrac{1}{p!}F_{(p)\;\mu_1\cdots\mu_p}F_{(p)}{}^{\mu_1\cdots\mu_p}\,, \tag{3.1}$$

The NSD IIB action written in forms reads

$$\begin{aligned} S_{IIB} &= \int d^{10}x\sqrt{|\hat{g}|}\;\Big[\hat{R}(\hat{g}) - \tfrac{1}{4}\,\mathrm{Tr}\Big(\partial_{\hat{\mu}}\hat{\mathcal{M}}\cdot\partial^{\hat{\mu}}\hat{\mathcal{M}}^{-1}\Big)\Big] \\ &\quad + \int_{10}\Big\{\tfrac{1}{2}\hat{\vec{H}}^T\hat{\mathcal{M}}^{-1}\,{}^{\star}\hat{\vec{H}} + \tfrac{1}{4}\hat{F}_{(5)}{}^{\star}\hat{F}_{(5)} + \tfrac{1}{4}\hat{F}_{(5)}\hat{\vec{B}}^T\eta\hat{\vec{H}}\Big\}\,, \end{aligned} \tag{3.2}$$

where we have defined

$$\left\{\begin{array}{rcl} \hat{\vec{H}} & = & d\hat{\vec{B}}\,, \\[2ex] \hat{F}_{(5)} & = & d\hat{D} - \tfrac{1}{2}\hat{\vec{B}}^T\eta\hat{\vec{H}}\,, \end{array}\right. \tag{3.3}$$

which are nothing else than the definitions in Eqs. (2.9), but written in terms of forms.

In order to follow through the above procedure, we start by gauging the $SL(2,\mathbb{R})$ symmetry. We introduce a covariant derivative through

$$\left\{\begin{array}{lcl} \partial_{\hat{\mu}}\hat{\mathcal{M}} & \rightarrow & \mathcal{D}_{\hat{\mu}}\hat{\mathcal{M}} \;=\; \partial_{\hat{\mu}}\hat{\mathcal{M}} + \hat{\mathcal{E}}_{\hat{\mu}}\hat{\mathcal{M}} + \hat{\mathcal{M}}\hat{\mathcal{E}}^{T}_{\hat{\mu}}\,, \\ \\ d\hat{\vec{B}} & \rightarrow & \mathcal{D}\hat{\vec{B}} \;=\; d\hat{\vec{B}} \,+\, \hat{\mathcal{E}}\wedge\hat{\vec{B}}\,, \end{array}\right. \tag{3.4}$$

and one finds that $\hat{\mathcal{E}}$ has to transform as a gauge field

$$\hat{\mathcal{E}} \rightarrow \Lambda^{-1}\hat{\mathcal{E}}\Lambda \,+\, \Lambda^{-1}d\Lambda\,. \tag{3.5}$$

Now, applying the same KK Ansatz for the metric as was used in the preceding section, one sees that the covariant derivatives on $\hat{\mathcal{M}}$ get transformed into, changing notation such that $\mathcal{E}$ is the constant matrix in the internal direction,

$$\left\{\begin{array}{lcl} \mathcal{D}_a\hat{\mathcal{M}} & = & \partial_a\mathcal{M} - A_{(1)a}\left[\mathcal{E}\mathcal{M} + \mathcal{M}\mathcal{E}^T\right]\,, \\ \mathcal{D}_y\mathcal{M} & = & K^{3/4}\left(\mathcal{E}\mathcal{M} + \mathcal{M}\mathcal{E}^T\right)\,. \end{array}\right. \tag{3.6}$$

Clearly $\mathcal{E}$ is going to be the mass matrix m. This then means that we can write down

$$\left\{\begin{array}{lcl} \mathrm{Tr}\left(\partial\hat{\mathcal{M}}\partial\hat{\mathcal{M}}^{-1}\right) & \rightarrow & \mathrm{Tr}\left(\partial\mathcal{M}\partial\mathcal{M}^{-1}\right) \\ & & +2A_{(1)\mu}\mathrm{Tr}\left[\mathcal{M}^{-1}\partial^{\mu}\mathcal{M}\left(\mathcal{M}^{-1}\mathcal{E}\mathcal{M} + \mathcal{E}^T\right)\right] \\ & & +2\left(K^{\frac{3}{2}} - A^2_{(1)}\right)\,\mathrm{Tr}\left(\mathcal{M}^{-1}\mathcal{E}\mathcal{M}\mathcal{E}^T + \mathcal{E}^2\right) \end{array}\right. \tag{3.7}$$

One will readily acknowledge that this is exactly the result found in Section 2.2.2 with $\mathcal{E} = m$.

Decomposing $\hat{\vec{B}}$ as

$$\hat{\vec{B}} \;=\; \vec{A}_{(2)} \;-\; \vec{A}_{(1)}d\underline{y}\,, \tag{3.8}$$

one finds that the reduction of $\hat{\vec{H}}$ leads to

$$\left\{\begin{array}{lcl} \hat{\vec{H}} & = & \vec{F}_{(3)} \;-\; K^{\frac{3}{4}}\vec{F}_{(2)}dy\,, \\ \vec{F}_{(2)} & = & d\vec{A}_{(1)} \;-\; \mathcal{E}\vec{A}_{(2)}\,, \\ \vec{F}_{(3)} & = & d\vec{A}_{(2)} \;+\; A_{(1)}\vec{F}_{(2)}\,. \end{array}\right. \tag{3.9}$$

This then allows us to reduce the $\hat{\vec{H}}$ term in the action as

$$\int_{10}\hat{\vec{H}}^T\mathcal{M}^{-1}\,{}^\star\hat{\vec{H}} \;=\; \int_9\left[K^{-\frac{3}{4}}\vec{F}^{\,T}_{(3)}\mathcal{M}^{-1}\,{}^\star\vec{F}_{(3)} - K^{\frac{3}{4}}\vec{F}^{\,T}_{(2)}\mathcal{M}^{-1}\,{}^\star\vec{F}_{(2)}\right]\,. \tag{3.10}$$

Doing the same thing on the 5-form field strength, we find that

$$\int_{10} \hat{F}_{(5)}{}^{\star}\hat{F}_{(5)} \;=\; \int_{9} \left[K^{-\frac{3}{4}}F_{(5)}{}^{\star}F_{(5)} - K^{\frac{3}{4}}F_{(4)}{}^{\star}F_{(4)}\right] \;, \tag{3.11}$$

where we have used

$$\left\{\begin{array}{rcl} \hat{D} &=& A_{(4)} \;-\; A_{(3)}d\underline{y} \;, \\ F_{(4)} &=& dA_{(3)} \;+\; \tfrac{1}{2}\vec{A}^{\,T}_{(1)}\eta\vec{F}_{(3)} \;-\; \tfrac{1}{2}\vec{A}^{\,T}_{(2)}\eta\vec{F}_{(2)} \;+\; \tfrac{1}{2}A_{(1)}\vec{A}^{\,T}_{(1)}\eta\vec{F}_{(2)} \;, \\ F_{(5)} &=& dA_{(4)} \;+\; A_{(1)}F_{(4)} \;+\; \tfrac{1}{2}A_{(1)}\vec{A}^{\,T}_{(2)}\eta\vec{F}_{(2)} \;-\; \tfrac{1}{2}\vec{A}^{\,T}_{(2)}\eta\vec{F}_{(3)} \;. \end{array}\right. \tag{3.12}$$

Now, reducing the CS-term and dualizing the $d=9$ 5-form field strength we end up with the following contribution to the $d=9$ action

$$\begin{array}{rcl} S_{(4)} &=& \int_9 \Big\{-\tfrac{1}{2}K^{\frac{3}{4}}F_{(4)}{}^{\star}F_{(4)} - \tfrac{1}{2}F_{(4)}F_{(4)}A_{(1)} + \tfrac{1}{2}F_{(4)}\vec{A}^{\,T}_{(2)}\eta\left(\vec{F}_{(3)} - A_{(1)}\vec{F}_{(2)}\right) \\ &+& \tfrac{1}{8}\left[\vec{A}^{\,T}_{(2)}\eta\vec{F}_{(2)} - A^{\,T}_{(1)}\eta\left(\vec{F}_{(3)} - A_{(1)}\vec{F}_{(2)}\right)\right]\vec{A}^{\,T}_{(2)}\eta\left(\vec{F}_{(3)} - A_{(1)}\vec{F}_{(2)}\right)\Big\} \;. \end{array} \tag{3.13}$$

Comparing the above results with the results in Eq. (2.49) one can see that both ways of reducing lead to the same thing.

3.1 Derivation of the massive transformations

Before the gauging, in $d=10$, we have the invariance

$$\delta\hat{\vec{B}} \;=\; d\hat{\vec{\mathcal{N}}} \;, \tag{3.14}$$

and we want to find the effect of these transformations after the gauging and the reduction: These will turn out to be related to *some of* the massive transformations.

When gauging the action, we have to covariantize the corresponding transformations. Since the $SL(2,\mathbb{R})$ acts on the $\hat{\vec{B}}$ fields, it is only natural to introduce the covariantized transformation rules

$$\delta\hat{\vec{B}} \;=\; d\hat{\vec{\mathcal{N}}} \;\rightarrow\; \delta\hat{\vec{B}} \;=\; \mathcal{D}\hat{\vec{\mathcal{N}}} \;=\; d\hat{\vec{\mathcal{N}}} \;+\; \hat{\mathcal{E}}\wedge\hat{\vec{\mathcal{N}}} \;, \tag{3.15}$$

under which the field strength for the $\hat{\vec{B}}$ field transforms as

$$\delta\hat{\vec{H}} \;=\; F(\hat{\mathcal{E}})\wedge\hat{\vec{\mathcal{N}}} \;, \tag{3.16}$$

where we have defined $F(\hat{\mathcal{E}}) = d\hat{\mathcal{E}} + \hat{\mathcal{E}}\wedge\hat{\mathcal{E}}$. This looks worse than it actually is: Since we take the gauge field to be constant and in one direction only, the field strength for the gauge field $\hat{\mathcal{E}}$ is identically zero, rendering the variation for $\hat{\vec{H}}$ nil.

Splitting the $\hat{\vec{B}}$ fields then as before, and defining

$$\hat{\vec{\mathcal{N}}} \;=\; \vec{\Sigma}_{(1)} \;-\; \vec{\Sigma}_{(0)} d\underline{y} \;, \tag{3.17}$$

one finds the following massive transformations

$$\left\{ \begin{array}{rcl} \delta \vec{A}_{(2)} & = & d\vec{\Sigma}_{(1)} \;, \\ \\ \delta \vec{A}_{(1)} & = & d\vec{\Sigma}_{(0)} \;+\; \mathcal{E}\vec{\Sigma}_{(1)} \,. \end{array} \right. \tag{3.18}$$

One can then see that the field strengths for the $d = 9$ fields $\vec{A}_{(2)}$ and $\vec{A}_{(1)}$ are indeed invariant under these transformations, and are $SL(2,\mathbb{R})$ invariant.

Under the $d = 10$ transformation $\delta\hat{\vec{B}} = d\hat{\vec{\mathcal{N}}}$ one finds that

$$\delta\hat{F}_{(5)} \;=\; d\delta\hat{D} - \tfrac{1}{2}(\delta\hat{\vec{B}})^T \eta \hat{\vec{H}} \;, \tag{3.19}$$

because $\hat{\vec{H}}$ is invariant. Now, using the facts

$$d\hat{\mathcal{E}} = 0 \;,\; \hat{\mathcal{E}} \wedge \hat{\mathcal{E}} = 0 \;,\; (\hat{\mathcal{E}} \wedge \hat{\vec{\mathcal{N}}})^T = -\hat{\vec{\mathcal{N}}}^T \wedge \hat{\mathcal{E}}^{\;T} \;,\; \hat{\mathcal{E}}^{\;T}\eta \;=\; -\eta\hat{\mathcal{E}} \;, \tag{3.20}$$

one finds that the variation reads

$$\delta\hat{F}_{(5)} \;=\; d\delta\hat{D} - \tfrac{1}{2} d\left(\hat{\vec{\mathcal{N}}}^{\;T}\eta\hat{\vec{H}}\right) \;. \tag{3.21}$$

This then means that iff

$$\delta\hat{D} \;=\; d\hat{\Delta}^{(3)} + \tfrac{1}{2}\hat{\vec{\mathcal{N}}}^{\;T}\eta\hat{\vec{H}} \;, \tag{3.22}$$

the 5-form field strength is invariant.

Dimensional reduction of the above transformation rule, leads to the variation rule for the 3-form, i.e.

$$\delta A_{(3)} \;=\; d\Delta^{(2)} + \tfrac{1}{2}\vec{\Sigma}_{(1)}^{\;T}\eta\vec{F}_{(2)} - \tfrac{1}{2}\vec{\Sigma}_{(0)}{}^{T}\eta\left[\vec{F}_{(3)} - A_{(1)}\vec{F}_{(2)}\right] \;. \tag{3.23}$$

Clearly, these transformations correspond to the non-χ transformations found in the preceding subsection.

4 11-Dimensional Origin of $N = 2, d = 9$ Massive Supergravities

In this Section we construct an 11-dimensional action which, upon dimensional reduction (zero-mode compactification) over a 2-torus gives the massive 9-dimensional type II supergravity action Eq. (2.49). In Section 2 it was important for us to keep $SL(2,\mathbb{R})$-covariance throughout the dimensional reduction and as a result we got a general action which describes a 3-parameter family of massive 9-dimensional type II supergravities. The three mass parameters transform in the adjoint representation of $SL(2,\mathbb{R})$ and thus, an $SL(2,\mathbb{R})$ transformation takes us from one member of the family (a supergravity theory) to another one.

Thus, in order to make contact with that result from an 11-dimensional (that is, from a type IIA/M-theoretical) starting point, it is important to have full control over the $SL(2,\mathbb{R}) \subset GL(2,\mathbb{R})$ symmetry that arises in the dimensional reduction in two dimensions. This symmetry in the type IIA side exactly corresponds to the S duality of the type IIB side [6, 12, 13, 14]. Thus, we will first reduce standard 11-dimensional supergravity making this symmetry manifest.

4.1 Compactification of 11-Dimensional Supergravity on T^2 and $Sl(2,\mathbb{R})$ Symmetry

The bosonic fields of $N = 1, d = 11$ supergravity [29] are the Elfbein and a 3-form potential

$$\left\{\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}}, \hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}}\right\} . \tag{4.1}$$

The field strength of the 3-form is

$$\hat{\hat{G}} = 4\partial\hat{\hat{C}} , \tag{4.2}$$

and is obviously invariant under the gauge transformations

$$\delta\hat{\hat{C}} = 3\partial\hat{\hat{\chi}} , \tag{4.3}$$

where $\hat{\hat{\chi}}$ is a 2-form. The action for these bosonic fields is

$$\hat{\hat{S}} = \int d^{11}x \sqrt{|\hat{\hat{g}}|} \left[\hat{\hat{R}} - \tfrac{1}{2\cdot 4!}\hat{\hat{G}}^2 - \tfrac{1}{6^4}\frac{1}{\sqrt{|\hat{\hat{g}}|}}\hat{\hat{\epsilon}}\partial\hat{\hat{C}}\partial\hat{\hat{C}}\hat{\hat{C}}\right] . \tag{4.4}$$

We have 2 mutually commuting Killing vectors $\{\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\}$ and use coordinates adapted to both of them: $\{\hat{\hat{x}}^{\hat{\hat{\mu}}}\} = \{x^\mu, x^m\}$ with $m = 9, 10$ and $x^9 = x, x^{10} = z$ and

$$\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\frac{\partial}{\partial\hat{\hat{x}}^{\hat{\hat{\mu}}}} = \frac{\partial}{\partial x^m} . \tag{4.5}$$

In these coordinates

$$\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\hat{\hat{k}}_{(n)}{}^{\hat{\hat{\nu}}}\hat{\hat{g}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}} = \hat{\hat{g}}_{mn} . \tag{4.6}$$

This is the internal space metric and it is in general non-diagonal, so the Killing vectors are not mutually orthogonal in general.

The standard KK Ansatz is[12]

$$\left(\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}}\right) = \begin{pmatrix} e_\mu{}^a & e_m{}^i A^{(m)}{}_\mu \\ & \\ 0 & e_m{}^i \end{pmatrix}, \qquad \left(\hat{\hat{e}}_{\hat{\hat{a}}}{}^{\hat{\hat{\mu}}}\right) = \begin{pmatrix} e_a{}^\mu & -A^{(m)}{}_a \\ & \\ 0 & e_i{}^m \end{pmatrix}, \tag{4.7}$$

where $A^{(m)}{}_a = e_a{}^\mu A^{(m)}{}_\mu$. For the metric, this means the following decomposition in 9-dimensional fields:

$$\begin{cases} \hat{\hat{g}}_{\mu\nu} = g_{\mu\nu} + G_{mn}A^{(m)}{}_\mu A^{(n)}{}_\nu \,, \\ \\ \hat{\hat{g}}_{\mu m} = G_{mn}A^{(n)}{}_\mu = \hat{\hat{k}}_{(m)\ \mu} \,, \\ \\ \hat{\hat{g}}_{mn} = G_{mn} = \hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\hat{\hat{k}}_{(n)\ \hat{\hat{\mu}}} \,. \end{cases} \tag{4.8}$$

The inverse relations are given in Appendix A.

From now on we will write the internal metric in matrix form and the two KK vectors in a column vector form:

$$G \equiv \begin{pmatrix} G_{\underline{xx}} & G_{\underline{xz}} \\ G_{\underline{zx}} & G_{\underline{zz}} \end{pmatrix}, \qquad \vec{A}_\mu \equiv \begin{pmatrix} A^{(\underline{x})}{}_\mu \\ \\ A^{(\underline{z})}{}_\mu \end{pmatrix}. \tag{4.9}$$

Under global transformations in the internal space

$$x^{m\,\prime} = \left(R^{-1\ T}\right)^m{}_n\, x^n + a^m \,, \qquad R \in GL(2,\mathbb{R}) \,, \tag{4.10}$$

objects with internal space indices transform as follows:

$$G' = RGR^T \,, \qquad \vec{A}'_\mu = (R^{-1})^T \vec{A}_\mu \,. \tag{4.11}$$

We know that $GL(2,\mathbb{R})$ can be decomposed in $SL(2,\mathbb{R}) \times \mathbb{R}^+ \times \mathbb{Z}_2$ and any matrix R can therefore be decomposed into

$$R = a\Lambda(\sigma^1)^\alpha \,, \quad \Lambda \in SL(2,\mathbb{R}) \,, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha = 0,1 \,, \quad a \in \mathbb{R}^+ \,. \tag{4.12}$$

The effect of a $\mathbb{Z}_2$ transformation σ^1 is the relabeling of the two internal coordinates and we will ignore it. Thus, we will focus on $GL(2,\mathbb{R})/\mathbb{Z}_2 \sim SL(2,\mathbb{R}) \times \mathbb{R}^+$. We want to separate fields that transform under the different factors. First we define the symmetric $SL(2,\mathbb{R})$ matrix[13]

[12] This is not exactly the standard KK Ansatz, which includes a rescaling of the lower-dimensional metric to end up in the Einstein conformal frame. We will perform the rescaling as a second step for pedagogical reasons.

[13] The minus sign is due to our mostly minus signature which makes the internal metric negative definite. We want $\mathcal{M}$ to be positive definite.

$$\mathcal{M} = -G/|\det\, G|^{1/2}\,, \tag{4.13}$$

and the scalar

$$K = |\det\, G|^{1/2}\,. \tag{4.14}$$

Now, under $SL(2,\mathbb{R})$ only $\mathcal{M}$ and $\vec{A}_\mu$ transform:

$$\mathcal{M}' = \Lambda\mathcal{M}\Lambda^T\,, \quad \vec{A}'_\mu = (\Lambda^{-1})^T\vec{A}_\mu\,, \tag{4.15}$$

that is, $\vec{A}_\mu$ transforms contravariantly, while under $\mathbb{R}^+$ rescalings only K and $\vec{A}_\mu$ transform:

$$K' = aK\,, \quad \vec{A}'_\mu = a\vec{A}_\mu\,. \tag{4.16}$$

It is convenient for our purposes to use a slightly different set of vector fields $\vec{A}_{(1)\ \mu}$ transforming covariantly under $SL(2,\mathbb{R})$, defined as follows:

$$\vec{A}_{(1)\ \mu} = \eta\vec{A}_\mu\,, \quad \vec{F}_{(2)\ \mu\nu} = 2\partial_{[\mu}\vec{A}_{(1)\ \nu]}\,, \quad \vec{A}'_{(1)\ \mu} = a\Lambda\vec{A}_{(1)\ \mu}\,. \tag{4.17}$$

Using the standard techniques, the above Elfbein Ansatz and rescaling the resulting 9-dimensional metric to the Einstein frame

$$g_{\mu\nu} = K^{-2/7}g_{E\ \mu\nu}\,, \tag{4.18}$$

one finds

$$\begin{aligned}\int d^{11}\hat{\hat{x}}\ \sqrt{|\hat{\hat{g}}|}\ \left[\ \hat{\hat{R}}\ \right] \ &= \ \int d^9x\ \sqrt{|g_E|}\ \Big[R_E + \tfrac{9}{14}\left(\partial\log K\right)^2 \\ &\quad +\tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{M}\mathcal{M}^{-1}\right)^2 - \tfrac{1}{4}K^{\frac{9}{7}}\vec{F}^{\ T}_{(2)}\mathcal{M}^{-1}\vec{F}_{(2)}\Big]\,.\end{aligned} \tag{4.19}$$

The 3-form term can be reduced along the same lines and we decompose the 11-dimensional 3-form potential into the 9-dimensional fields $A_{(3)\ \mu\nu\rho}, \vec{A}_{(2)\ \mu\nu}$ and $A_{(1)\ \mu}$ as follows:

$$\left\{\begin{aligned} \hat{\hat{C}}_{\mu\nu\rho} \ &= \ A_{(3)\ \mu\nu\rho} + \tfrac{3}{2}\vec{A}^{\ T}_{(1)\ [\mu}\eta\vec{A}_{(2)\ \nu\rho]} + 3A_{(1)\ [\mu}\vec{A}^{\ T}_{(1)\ \nu}\eta\vec{A}_{(1)\rho]}\,, \\ \begin{pmatrix}\hat{\hat{C}}_{\mu\nu\underline{x}} \\ \hat{\hat{C}}_{\mu\nu\underline{z}}\end{pmatrix} \ &= \ \vec{A}_{(2)\ \mu\nu} - 2A_{(1)\ [\mu}\vec{A}_{(1)\ \nu]}\,, \\ \begin{pmatrix} 0 & \hat{\hat{C}}_{\mu\underline{xz}} \\ \hat{\hat{C}}_{\mu\underline{zx}} & 0\end{pmatrix} \ &= \ +\eta A_{(1)\ \mu}\,, \end{aligned}\right. \tag{4.20}$$

The corresponding 9-dimensional field strengths $F_{(4)}, \vec{F}_{(3)}$ and $F_{(2)}$ are defined exactly by the massless limit of Eq. (2.42). The relation with the 11-dimensional field strength $\hat{\hat{G}}$ is

$$\left\{\begin{array}{rcl}
\hat{\hat{G}}_{\mu\nu\rho\sigma} & = & F_{(4)\ \mu\nu\rho\sigma} - 4\vec{A}^{\,T}_{(1)\ [\mu}\eta\vec{F}_{(3)\ \nu\rho\sigma]} \\
& & +5\vec{A}^{\,T}_{(1)\ [\mu}\eta\vec{A}_{(1)\ \nu}F_{(2)\ \rho\sigma]}\,, \\
\begin{pmatrix} \hat{\hat{G}}_{\mu\nu\rho\underline{x}} \\ \hat{\hat{G}}_{\mu\nu\rho\underline{z}} \end{pmatrix} & = & \vec{F}_{(3)\ \mu\nu\rho} - 3\vec{A}_{(1)\ [\mu}F_{(2)\ \nu\rho]}\,, \\
\begin{pmatrix} 0 & \hat{\hat{G}}_{\mu\nu\underline{xz}} \\ \hat{\hat{G}}_{\mu\nu\underline{zx}} & 0 \end{pmatrix} & = & \eta F_{(2)\ \mu\nu}\,.
\end{array}\right. \tag{4.21}$$

This allows us to decompose the kinetic term as follows:

$$\sqrt{|\hat{\hat{g}}|}\tfrac{-1}{2\cdot 4!}\hat{\hat{G}}^2 = \sqrt{|g_E|}\left\{\tfrac{-1}{2\cdot 4!}K^{6/7}F^2_{(4)} + \tfrac{1}{2\cdot 3!}K^{-3/7}\vec{F}^{\,T}_{(3)}\mathcal{M}^{-1}\vec{F}_{(3)} - \tfrac{1}{4}K^{-12/7}F^2_{(2)}\right\}\,. \tag{4.22}$$

and the topological term as follows:

$$\begin{array}{rcl}
\frac{1}{(144)^2}\,\hat{\hat{\epsilon}}\hat{\hat{G}}\hat{\hat{G}}\hat{\hat{C}} & = & \frac{1}{3^2\cdot 2^8}\epsilon\epsilon^{mn}\left\{\hat{\hat{G}}\hat{\hat{G}}\hat{\hat{C}}_{mn} + 4\hat{\hat{G}}\hat{\hat{G}}_m\hat{\hat{C}}_n\right\} \\
& = & \frac{1}{3^2\cdot 2^7}\epsilon\left[F_{(4)} - 4\vec{A}_{(1)}{}^T\eta\vec{F}_{(3)} + 6\vec{A}_{(1)}{}^T\eta\vec{A}_{(1)}F_{(2)}\right]\times \\
& & \times\left\{\left[F_{(4)} - 4\vec{A}_{(1)}{}^T\eta\vec{F}_{(3)} + 6\vec{A}_{(1)}{}^T\eta\vec{A}_{(1)}F_{(2)}\right]A_{(1)}\right. \\
& & \left.+2\left[\vec{F}_{(3)} - 3\vec{A}_{(1)}F_{(2)}\right]^T\eta\left[\vec{A}_{(2)} + 2\vec{A}_{(1)}A_{(1)}\right]\right\}\,.
\end{array} \tag{4.23}$$

Putting all our partial results together, Eqs. (4.19,4.22,4.23), we arrive at the action of type II 9-dimensional supergravity in Einstein frame, Eq. (2.49), which we obtained through generalized dimensional reduction of the 10-dimensional type IIB theory with the mass matrix set to zero [6].

The fact that upon dimensional reduction the type IIA and type IIB *supergravity theories* are identical in nine dimensions is nothing but the manifestation at the level of the massless modes of the T duality existing between the type IIA and type IIB *superstring theories* when they are compactified in circles of dual radii [30, 31].

There are four important points we would like to stress:

1. There is no "hidden symmetry" of the 9-dimensional type II theory corresponding to this T duality.

2. To obtain two *identical* actions it is crucial that the two topological terms come with the same global sign. In the M/type IIA side the sign can be changed by the 11-dimensional transformation $\hat{\hat{C}} \to -\hat{\hat{C}}$ which is not a symmetry. In the type IIB side, flipping the sign of the 4-form $\hat{D}$ does not work because it changes the definition of its field strength. Changing the signs of $\hat{D}$ and, say, $\hat{B}^{(1)}$ leaves $\hat{F}$ invariant but also leaves invariant the topological term. Thus, at first sight, there seems to be no IIB-side version of this rather trivial M/IIA-side transformation.

   It is, however, easy to see that the sign of the topological term in the NSD 10-dimensional type IIB action is directly related to the self-duality of the 5-form field strength. Had we considered an anti-self-dual 5-form the sign would have been exactly the opposite in ten and nine dimensions. The (anti-) self-duality of the 5-form is related to the chirality of the theory.

   The picture that emerges is therefore the following: There are two (otherwise equivalent) 11-dimensional supergravity theories and two 10-dimensional type IIA theories that differ only in the sign of the action's topological term. Upon dimensional reduction to nine dimensions they are related to the two type IIB theories of opposite chiralities.

   In the decompactification limit, each of these two 9-dimensional (and, thus, non-chiral) theories knows to which chiral 10-dimensional type IIB theory it should decompactify.

3. The above observation solves in part the puzzle found in Ref. [32] where it was argued that approximately half of all extreme black holes are not supersymmetric in type II theories. Clearly, those which are not supersymmetric in one of the 11-dimensional supergravities are supersymmetric in the 11-dimensional supergravity with the sign of the 3-form $\hat{\hat{C}}$ reversed. As suggested also in Ref. [33], the whole picture begs for both 11-dimensional supergravities to be integrated into a higher-dimensional supergravity from which also the type IIB would be derivable, perhaps one of those with the algebras studied in Ref. [34]. (This argument is completely different from the one in Ref. [35] and, in fact, it is in disagreement with it).

4. The fact that the two theories (A and B) are identical allows us to relate the 10-dimensional fields of the two type II theories. This relation provides a generalization of Buscher's T duality rules [3]. These type II Buscher rules were found in Ref. [6] and they are determined again in Appendices A, B and C in our (more systematic) conventions and extended to the massive case at hands.

4.2 $Sl(2,\mathbb{R})$-Covariant Massive 11-Dimensional Supergravity

So much for the massless case. Now, it is clear that the picture seems to break down whenever the mass matrix does not vanish. In Ref. [8] the particular case with mass matrix with $m^1 = 0, m^2 = m^3 = m$

$$m_{\rm BRGPT} = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \tag{4.24}$$

was considered. As will be discussed in Section 5 this particular choice of mass matrix corresponds to compactification of the type IIB on a background with different species of 7-branes. Since the T dual of a D-7-brane in a direction orthogonal to its worldvolume is a type IIA D-8-brane, one expects the theory with mass matrix $m_{\rm BRGPT}$ to correspond to the type IIA theory on a background with D-8-branes.

While it is not possible to write the 10-dimensional type IIB theory in presence of D-7-branes in a covariant fashion (there is dependence on the compactifying coordinate y) it is possible to write in a covariant fashion the action for the type IIA theory in presence of D-8-branes. As was first first realized in Ref. [10], this theory has long been known as Romans' massive type IIA supergravity [9]. The precise identification, leading to a further generalization of Buscher's rules was carried out in Ref. [8]. We stress that these T duality rules are essentially identical to the original type II T duality rules of Ref. [6] but are deformed in a y-dependent fashion in the type IIB side of the equations.

Our task in the remainder of this Section will be to generalize the results of Ref. [8]. It is clear from the setting that this generalization amounts to its $SL(2,\mathbb{R})$-covariantization: We start from the compactification of the type IIB theory on a background containing D-7-branes and their S duals and, after T duality, we expect to find a type IIA theory on a background of the T duals of D-7-branes and their S duals. We will not repeat here the discussion of the Introduction where we concluded that we must look for a non-covariant generalization of Romans' type IIA supergravity.

As a matter of fact, it is easier to generalize the 11-dimensional theory that gives Romans', given in Ref. [18]. From our point of view this theory would correspond to 11-dimensional supergravity with a KK-9M-brane in the background. To find in 9-dimensions an $SL(2,\mathbb{R})$-covariant result we must consider a theory describing 11-dimensional supergravity with two KK-9M-branes in the background.

In what follows we will construct such a theory along the same lines as Ref. [18] and show that it gives the massive 9-dimensional type II theory constructed in Section 2.

Since each KK-9M-brane is associated to a Killing vector we assume the presence of the two mutually commuting Killing vectors of the previous Section and also assume that the Lie derivatives of all fields with respect to both of them vanishes.

Next, we define the 11-dimensional massive transformations. For a general tensor, except for $\hat{\hat{C}}$ whose transformation law will be defined below, they are

$$\delta_{\hat{\hat{\chi}}} L_{\hat{\hat{\mu}}_1\cdots\hat{\hat{\mu}}_r} = \hat{\hat{\lambda}}^{(n)}{}_{\hat{\hat{\mu}}_1} \hat{\hat{k}}_{(n)}{}^{\hat{\hat{\nu}}} \hat{\hat{L}}_{\hat{\hat{\nu}}\hat{\hat{\mu}}_2\cdots\hat{\hat{\mu}}_r} + \ldots + \hat{\hat{\lambda}}^{(n)}{}_{\hat{\hat{\mu}}_r} \hat{\hat{k}}_{(n)}{}^{\hat{\hat{\nu}}} \hat{\hat{L}}_{\hat{\hat{\mu}}_1\cdots\hat{\hat{\mu}}_{r-1}\hat{\hat{\nu}}}\,, \tag{4.25}$$

where we have defined

$$\hat{\hat{\lambda}}^{(n)} \equiv -i_{\hat{\hat{k}}_{(m)}}\hat{\hat{\chi}}Q^{nm}\,, \qquad Q^{nm} = \left(m^T\eta\right)^{mn} = \tfrac{1}{2}\left(\begin{array}{cc} -(m^2+m^3) & m^1 \\ & \\ m^1 & m^2-m^3 \end{array}\right)\,. \tag{4.26}$$

The contraction of a space tensor with the Killing vectors will bear an $SL(2,\mathbb{R})$ index: The extension of the above rule for incorporating $SL(2,\mathbb{R})$ indices is found by defining the inclusion to commute with the massive transformations.

In particular we find that the 11-dimensional metric and r-forms $\hat{\hat{S}}$ transform as

$$\left\{\begin{array}{rcl} \delta_{\hat{\hat{\chi}}}\hat{\hat{g}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}} & = & 2\hat{\hat{\lambda}}^{(n)}{}_{(\hat{\hat{\mu}}}\,\hat{\hat{k}}_{(n)}{}^{\hat{\hat{\rho}}}\hat{\hat{g}}_{\hat{\hat{\nu}})\hat{\hat{\rho}}}\,, \\ & & \\ \delta_{\hat{\hat{\chi}}}\hat{\hat{S}}_{\hat{\hat{\mu}}_1\cdots\hat{\hat{\mu}}_r} & = & (-)^{r-1}r\hat{\hat{\lambda}}^{(n)}{}_{[\hat{\hat{\mu}}_1}\,\hat{\hat{k}}_{(n)}{}^{\hat{\hat{\rho}}}\hat{\hat{S}}_{\hat{\hat{\mu}}_2\cdots\hat{\hat{\mu}}_r]\hat{\hat{\rho}}}\,. \end{array}\right. \tag{4.27}$$

Observe that these rules imply that

$$\left\{\begin{array}{rcl} \delta_{\hat{\hat{\chi}}}\sqrt{|\hat{\hat{g}}|} & = & 0\,, \\ & & \\ \delta_{\hat{\hat{\chi}}}\hat{\hat{S}}^2 & = & 0\,, \end{array}\right. \tag{4.28}$$

where the latter holds due to the fact that also the metric varies under the massive transformations, and the former holds due to the fact that the matrix $Q = m^T\eta$ is symmetric.

The 3-form field $\hat{\hat{C}}$ is going to play the role of a connection-field with respect to the massive transformations and, as such, does not transform covariantly

$$\delta_{\hat{\hat{\chi}}}\hat{\hat{C}} \;=\; d\hat{\hat{\chi}} \;+\; \hat{\hat{\lambda}}^{(n)}\wedge\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)\,. \tag{4.29}$$

The generalization of the field strength for $\hat{\hat{C}}$, denoted as before by $\hat{\hat{G}}$, is then found by requiring that the field strength does transform covariantly. One can see that this implies that

$$\hat{\hat{G}} \;=\; d\hat{\hat{C}} \;-\; \tfrac{1}{2}\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)Q^{nm}\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)\,. \tag{4.30}$$

Comparing this with a torsionful covariant derivative acting on a 3-form, one sees that the above equation states that the massive transformations induce a torsion term in our spacetime connection. This then means that if we want our $d = 11$ theory to be invariant under the massive transformations, we have to define our theory in terms of the torsionful connection.

The torsion we need is given by

$$\hat{\hat{T}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}}{}^{\hat{\hat{\rho}}} = -\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)_{\hat{\hat{\mu}}\hat{\hat{\nu}}}Q^{nm}\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\rho}}}\,. \tag{4.31}$$

The torsionful connection $\hat{\hat{\Omega}}$ is then defined in the standard way, by adding the so-called contorsion-torsion tensor,

$$\hat{\hat{K}}_{\hat{\hat{a}}\hat{\hat{b}}\hat{\hat{c}}} = \tfrac{1}{2}\left(\hat{\hat{T}}_{\hat{\hat{a}}\hat{\hat{c}}\hat{\hat{b}}} + \hat{\hat{T}}_{\hat{\hat{b}}\hat{\hat{c}}\hat{\hat{a}}} - \hat{\hat{T}}_{\hat{\hat{a}}\hat{\hat{b}}\hat{\hat{c}}}\right) , \tag{4.32}$$

to the Levi-Cività connection $\hat{\hat{\omega}}$, i.e.

$$\hat{\hat{\Omega}}_{\hat{\hat{a}}}{}^{\hat{\hat{b}}\hat{\hat{c}}} = \hat{\hat{\omega}}_{\hat{\hat{a}}}{}^{\hat{\hat{b}}\hat{\hat{c}}} + \hat{\hat{K}}_{\hat{\hat{a}}}{}^{\hat{\hat{b}}\hat{\hat{c}}} . \tag{4.33}$$

From the above equation we can obtain the non-vanishing components of the torsion written directly in 9-dimensional Lorentz coordinates for future use

$$\left\{\begin{array}{rcl} \hat{\hat{T}}_{abi} & = & -A_{(2)(n)ab}\eta^{np}m_p{}^q e_{qi} , \\ \\ \hat{\hat{T}}_{aij} & = & A_{(1)a}e_i{}^p m_p{}^q e_{qi} . \end{array}\right. \tag{4.34}$$

Having all this, one can see that the 11-dimensional theory invariant under the massive transformation reads[14]

$$\begin{aligned} \hat{\hat{S}} \;=\; & \int d^{11}\hat{\hat{x}}\sqrt{|\hat{\hat{g}}|}\, \Big\{\hat{\hat{R}}\left(\hat{\hat{\Omega}}\right) + \left(d\hat{\hat{k}}_{(n)}\right)_{\hat{\hat{\mu}}\hat{\hat{\nu}}} Q^{nm}\left(i_{\hat{\hat{k}}_{(m)}}\hat{\hat{C}}\right)^{\hat{\hat{\mu}}\hat{\hat{\nu}}} - \tfrac{1}{2\cdot 4!}\hat{\hat{G}}^2 \\ & -2\hat{\hat{K}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}}\hat{\hat{K}}^{\hat{\hat{\nu}}\hat{\hat{\rho}}\hat{\hat{\mu}}} + \tfrac{1}{2}\left(\hat{\hat{k}}_{(n)\ \hat{\hat{\mu}}}Q^{nm}\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\right)^2 - \left(\hat{\hat{k}}_{(n)\ \hat{\hat{\mu}}}Q^{nm}\hat{\hat{k}}_{(m)\ \hat{\hat{\nu}}}\right)^2 \\ & -\tfrac{1}{6^4}\frac{\hat{\hat{\epsilon}}}{\sqrt{|\hat{\hat{g}}|}}\Big\{\partial\hat{\hat{C}}\partial\hat{\hat{C}}\hat{\hat{C}} - \tfrac{9}{8}\partial\hat{\hat{C}}\hat{\hat{C}}\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)Q^{nm}\left(i_{\hat{\hat{k}}_{(m)}}\hat{\hat{C}}\right) \\ & +\tfrac{27}{80}\hat{\hat{C}}\left[\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)Q^{nm}\left(i_{\hat{\hat{k}}_{(m)}}\hat{\hat{C}}\right)\right]^2\Big\}\Big\} , \end{aligned} \tag{4.35}$$

For the dimensional reduction of the above theory, the fields will be split in the same way as in the preceding subsection; The only thing that changes, is the torsion part of the connection and some terms in the 11-dimensional Chern-Simons term.

Let us first consider the reduction of the curvature term, evaluated using the connection in Eq. (4.33). Using Palatini's identity for torsionful connections

$$\begin{aligned} \int_d \sqrt{|g|}\, e^{-2\phi}\, R(\Omega) \;=\; & -\int_d \sqrt{|g|}\, e^{-2\phi}\left\{\Omega_b{}^{ba}\Omega_c{}^c{}_a + \Omega_a{}^{bc}\Omega_{bc}{}^a + 4\Omega_b{}^{ba}\partial_a\phi \right. \\ & \left. -2\Omega_b{}^{ba}K_c{}^c{}_a - 2\Omega_a{}^{bc}K_{bc}{}^a\right\} , \end{aligned} \tag{4.36}$$

the facts

[14]Note that the cosmological constant part is, apart from correspondence with the massive $d = 9$ theory, arbitrary. However, supersymmetry should completely determine it.

$$\begin{cases} \hat{\hat{K}}_{\hat{\hat{a}}}{}^{\hat{\hat{a}}b} & = & A^b_{(1)}\hat{\hat{\eta}}^{ij} e_i{}^m e_{nj} m_m{}^n \; = \; A^b_{(1)} \mathrm{Tr}\,(m) \; = \; 0\,, \\ & & \\ \hat{\hat{K}}_{\hat{\hat{a}}}{}^{\hat{\hat{a}}i} & = & 0\,, \end{cases} \tag{4.37}$$

and the fact that the second term in Eq. (4.35) annihilates the ΩK-terms whilst applying Palatini's identity to the case at hand, one can write

$$\begin{aligned} \int d^{11}\hat{\hat{x}}\sqrt{|\hat{\hat{g}}|}\left\{\hat{\hat{R}}\left(\hat{\hat{\Omega}}\right) + \left(d\hat{\hat{k}}_{(n)}\right)_{\hat{\hat{\mu}}\hat{\hat{\nu}}} Q^{nm}\left(i_{\hat{\hat{k}}_{(m)}}\hat{\hat{C}}\right)^{\hat{\hat{\mu}}\hat{\hat{\nu}}} - 2\hat{\hat{K}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\kappa}}}\hat{\hat{K}}_{\hat{\hat{\nu}}\hat{\hat{\kappa}}\hat{\hat{\mu}}}\right\} \; = \\ = \int d^9x\sqrt{|g|}\; K\left[R(g) - (\partial\log K)^2 \; + \; \tfrac{1}{4}\left(F_{iab} + \hat{T}_{abi}\right)^2\right. \\ \left. +\tfrac{1}{4}\left(e_i{}^n e_j{}^m\partial_a G_{nm} \; + \; 2\hat{T}_{a(ij)}\right)^2\right) \;. \end{aligned} \tag{4.38}$$

Using now our previous partial results Eqs. (4.13,4.17,4.34) and rescaling to the Einstein frame, Eq. (4.18), this can be written as

$$= \int d^9x\sqrt{|g_E|}\;\left[R(g_E) + \tfrac{9}{14}(\partial\log K)^2 \; + \; \tfrac{1}{4}\mathrm{Tr}\,(\mathcal{D}\mathcal{M}\mathcal{M}^{-1})^2 - \tfrac{1}{4}K^{9/7}\vec{F}^{\,T}_{(2)}\mathcal{M}^{-1}\vec{F}_{(2)}\right]\,, \tag{4.39}$$

where the field strengths and covariant derivative are the same as the ones used in Section 2.

The cosmological constant part is readily reduced by using the well-known identity

$$\eta^{mn}\eta^{pq} \;=\; -\eta^{np}\eta^{mq} \;-\; \eta^{pm}\eta^{nq}\,, \tag{4.40}$$

and it follows that

$$\begin{aligned} \tfrac{1}{2}\int d^{11}\hat{\hat{x}}\sqrt{|\hat{\hat{g}}|}\;\left[\left(\hat{\hat{k}}_{(n)\;\hat{\hat{\mu}}}Q^{nm}\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\right)^2 - \left(\hat{\hat{k}}_{(n)\;\hat{\hat{\mu}}}Q^{nm}\hat{\hat{k}}_{(m)\;\hat{\hat{\nu}}}\right)^2\right] \;= \\ = -\tfrac{1}{2}\int d^9x\sqrt{|g_E|}\;K^{12/7}\mathrm{Tr}\left(m^2 \;+\; \mathcal{M}m\mathcal{M}^{-1}m^T\right)\;, \end{aligned} \tag{4.41}$$

which is just the result obtained in the $d = 9$ theory.

The effect of the torsion included in definition (4.30), can readily be seen to promote the field strengths to their massive equivalents Eq. (2.42). As such, it will be no surprise at all to see that

$$\begin{aligned} \int_{11} -\tfrac{1}{2}\hat{\hat{G}}{}^\star\hat{\hat{G}} \;\;=\;\; \int_9\Big\{-\tfrac{1}{2}K^{6/7}F_{(4)}{}^\star F_{(4)} \;+\; \tfrac{1}{2}K^{-3/7}\vec{F}^{\,T}_{(3)}\mathcal{M}^{-1\star}\vec{F}_{(3)} \\ -\tfrac{1}{2}K^{-12/7}F_{(2)}{}^\star F_{(2)}\Big\}\;. \end{aligned} \tag{4.42}$$

From the fact that we do not change the decomposition of the fields while doing the reduction, it is clear that the $d\hat{C}d\hat{C}\hat{C}$ will lead to the same result as in Eq. (4.23). The other terms can easily be seen to result in

$$
\begin{aligned}
\tfrac{1}{6^4}\int_{11}\hat{\hat{\epsilon}}\,\tfrac{9}{8}\partial\hat{\hat{C}}\hat{\hat{C}}\left[\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)Q^{nm}\left(i_{\hat{\hat{k}}_{(m)}}\hat{\hat{C}}\right)\right] &= \tfrac{1}{3^2 2^7}\int_9 \epsilon\,\Big[6F_{(4)}\,\vec{A}^T_{(2)}Q\vec{A}_{(2)}\,A_{(1)} \\
&\quad -9\left(\vec{A}^T_{(2)}Q\vec{A}_{(2)}\right)\left(\vec{A}^T_{(2)}\eta\partial\vec{A}_{(2)}\right)\Big]\,, \qquad (4.43) \\
\tfrac{1}{6^4}\int_{11}\hat{\hat{\epsilon}}\,\tfrac{27}{80}\hat{\hat{C}}\left[\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)Q^{nm}\left(i_{k_{(m)}}\hat{\hat{C}}\right)\right]^2 &= \tfrac{1}{3^2 2^7}\int_9 \epsilon\,9\left(\vec{A}^T_{(2)}Q\vec{A}_{(2)}\right)^2 A_{(1)}\,.
\end{aligned}
$$

Adding the above equations to Eq. (4.23) we find that the effect of the torsion is, once again, precisely to turn the massless CS term, into the massive CS term of the massive 9-dimensional type II theory we got by generalized dimensional reduction of the type IIB theory. Thus, we have achieved our second goal.

The T duality rules that one can immediately deduce from this relation between 10-dimensional theories will be worked out in the Appendices.

5 7-Branes

In this Section we want to identify the 10-dimensional background of the type IIB theory that produces the masses of the 9-dimensional theory. The T dual background will be dealt with in Section 6.

S duality is (believed to be) a fundamental non-perturbative symmetry of type IIB string theory. This implies that the full spectrum of the theory has to be S duality-invariant and thus all the states can be organized in $SL(2,\mathbb{Z})$ multiplets. Thus, bound states of q fundamental strings and p D-strings, known as pq-strings, transform as doublets under $SL(2,\mathbb{Z})$. A general solution describing all possible pq-strings was constructed in Ref. [36] and a dual general solution describing all possible pq-5-branes was recently constructed in Ref. [37]. The D-3-brane, being self-dual, is an $SL(2,\mathbb{Z})$ singlet. The situation for D-9-branes and D-instantons is unclear, although one expects to have D-9-brane solutions which only differ in the constant value of the dilaton.

It is commonly accepted that there are bound states of p D-7-branes and q NS-NS 7-branes (that we will call Q-7-branes) which transform as doublets. As we are going to see, this is not so clear and we will argue that 7-brane states transform as triplets. We will relate the monodromy matrices of massive 9-dimensional type II supergravity and these 7-brane triplets, showing again in this way that the presence of a background of 7-branes is the origin of the masses.

5.1 Point-Like (in Transverse Space) 7-Branes

The extreme D-7-brane solution in the string frame is

$$\left\{\begin{array}{rcl} ds^2 & = & H_{D7}^{-1/2}\,[dt^2 - d\vec{y}_7{}^2] - H_{D7}^{1/2} d\vec{x}_2{}^2\,, \\ e^{-2(\hat{\varphi}-\varphi_0)} & = & H_{D7}^2\,, \\ \hat{C}^{(8)}{}_{ty^1\cdots y^7} & = & \pm e^{-\hat{\varphi}_0} H_{D7}^{-1}\,, \end{array}\right. \tag{5.1}$$

where $\vec{y}_7 = (y_7^1, y_7^2, \ldots, y_7^7)$ are the worldvolume coordinates and $\vec{x}_2 = (x_2^1, x_2^2)$ are the coordinates of the 2-dimensional transverse space. Any function H_{D7} harmonic in the transverse space provides a D-7-brane-type solution. A harmonic function H_{D7} with a single point-like singularity

$$\partial_{x_2^i}\partial_{x_2^i} H_{D7} = 2\pi h_{D7}\delta^{(2)}(\vec{x}_2)\,, \tag{5.2}$$

describes a single D-7-brane placed at $\vec{x}_2 = 0$. The positive constant h_{D7} is proportional to the D-7-brane charge and mass and later on we will determine the precise relation between them. The two possible signs of the charge are taken care of by the $\pm$ in $\hat{C}^{(8)}$. The standard solution in $\mathbb{R}^2$ to the above equation is (the additive constant is arbitrary and momentarily we set to zero)

$$H_{D7} = h_{D7}\log|\vec{x}_2|\,. \tag{5.3}$$

The 8-form potential $\hat{C}^{(8)}$ is nothing but the dual of the RR scalar $\hat{C}^{(0)}$ that occurs in the type IIB theory (i.e. their field strengths are each other's Hodge dual $\hat{G}^{(1)} = {}^\star\hat{G}^{(9)}$). This dualization can only be done "on shell", i.e. using at the same time $\hat{C}^{(0)}$ and $\hat{C}^{(8)}$ because $\hat{C}^{(0)}$ occurs explicitly in the type IIB action. This gives the standard form of $\hat{G}^{(9)}$ suggested in Refs. [25, 26]. If we ignore all other fields apart from $\hat{\lambda}$ both dualizations are equivalent. Using this relation we find

$$\partial_i \hat{C}^{(0)} = \pm e^{-\hat{\varphi}_0}\epsilon_{ij}\partial_j H_{D7}\,, \tag{5.4}$$

and we can rewrite the solution in terms of just the metric and the two real scalars $\hat{C}^{(0)}, e^{-\hat{\varphi}}$ that we combine into the single complex scalar $\hat{\lambda} = \hat{C}^{(0)} + ie^{-\hat{\varphi}}$. For the single D-7-brane we find

$$\hat{\lambda} = \left\{\begin{array}{l} ie^{-\hat{\varphi}_0} h_{D7}\log\omega\,, \\ ie^{-\hat{\varphi}_0} h_{D7}\log\overline{\omega}\,, \end{array}\right. \qquad \omega = x_2^1 + ix_2^2\,, \tag{5.5}$$

for the upper and lower signs respectively.

The charge of a D-7-brane is just, with our normalizations (in the string frame)

$$p = \oint_\gamma {}^\star\hat{G}^{(9)} = \oint_\gamma \hat{G}^{(1)} = \oint_\gamma d\hat{C}^{(0)} = \Re\mathrm{e}\oint_\gamma d\hat{\lambda}\,. \tag{5.6}$$

The contour γ is any circle around the point in the transverse space. Using the residue theorem we find for our case that the imaginary part of the integral is zero and

$$p = \mp 2\pi e^{-\hat{\varphi}_0} h_{D7}\,, \tag{5.7}$$

so the solution indeed describes an anti-D-7-brane (upper sign, $\hat{\lambda} = \hat{\lambda}(\omega)$ a holomorphic function of ω) or D-7-brane (lower sign, $\hat{\lambda} = \hat{\lambda}(\overline{\omega})$ a holomorphic function of $\overline{\omega}$) for

$$h_{D7} = \frac{e^{\hat{\varphi}_0}}{2\pi}\,. \tag{5.8}$$

We stress that the transformation that takes us from the D-7-brane to the anti-D-7-brane with opposite RR charge is

$$\hat{\lambda}_{(p)} \to \hat{\lambda}_{(-p)} = -\overline{\hat{\lambda}_{(p)}}\,, \tag{5.9}$$

and it is not an $SL(2,\mathbb{R})$ transformation.

We have just associated the charge of the D-7-brane to the monodromy properties of the anti-holomorphic function $\hat{\lambda}(\overline{\omega})$: If we place at the origin a D-7-brane of unit charge, described by

$$\hat{\lambda}_{(p=1)} = -\frac{1}{2\pi i}\log\overline{\omega}\,, \tag{5.10}$$

and travel once along the path $\gamma(\xi)\,,\ \xi\in[0,1]$, around the origin

$$\begin{aligned} \hat{\lambda}_{(p=1)}[\gamma(1)] &= \hat{\lambda}_{(p=1)}[\gamma(0)] + 1 = \left(M_{(p=1)}\hat{\lambda}_{(p=1)}\right)[\gamma(0)]\,, \\ M_{(p=1)} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = T\,, \end{aligned} \tag{5.11}$$

where $M_{(p=1)}$ is the $SL(2,\mathbb{Z})$ monodromy matrix characterizing the 7-brane with charge $p=1$. One can then apply $SL(2,\mathbb{Z})$ transformations Λ to generate other solutions as done in Ref. [38]. Clearly, the monodromy matrix transforms in the adjoint representation

$$M' = \Lambda M \Lambda^{-1}\,. \tag{5.12}$$

Now, it is usually assumed that there are bound states of two kinds of 7-branes (*pq*-branes) transforming as doublets under $SL(2,\mathbb{Z})$. In particular, the charge vector of *pq*-7-branes transforms *covariantly* under $SL(2,\mathbb{Z})$, that is

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \Lambda \begin{pmatrix} p \\ q \end{pmatrix}\,. \tag{5.13}$$

The charge vector of *pq*-strings transforms contravariantly [36], that is

$$(p'\ q') = (p\ q)\Lambda^{-1}\,, \tag{5.14}$$

and so does the charge vector of *pq*-5-branes [37]. Using the above transformation law, one can generate, starting from the $(p=1) \equiv (1,0)$ other charge vectors using the $SL(2,\mathbb{Z})$ matrix $\Lambda_{(p,q)}$

$$\Lambda_{(p,q)} = \begin{pmatrix} p & b \\ q & d \end{pmatrix}, \qquad \Lambda_{(p,q)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}. \tag{5.15}$$

With the same transformation we generate the supergravity solution describing the pq-7-brane with those charges. The monodromy matrix that characterizes this solution is

$$M_{(p,q)} = \Lambda_{(p,q)} M_{(1,0)} \Lambda^{-1}_{(p,q)} = \begin{pmatrix} 1-pq & p^2 \\ & \\ -q^2 & 1+pq \end{pmatrix}. \tag{5.16}$$

Clearly not any pair (p,q) can be generated in this way from $(1,0)$. p and q cannot be even at the same time, to start with. According to the standard lore of S duality p and q have to be coprime in order to correspond to stable bound states, and thus this first objection does not seem serious. Still, there is no proof that all pairs corresponding to stable states can be generated in this way.

A second problem is that this is not (by far) the most general $SL(2,\mathbb{Z})$ matrix. Thus, given a certain monodromy matrix we cannot in general determine to which (p,q) state it corresponds.

But there is a more serious problem: We saw in Eq. (5.9) that the transformation that takes us from the $(1,0)$ state to the $(-1,0)$ state is not an $SL(2,\mathbb{Z})$ transformation. However, if the rule Eq. (5.13) is true the transformation $-\mathbb{I}_{2\times 2}$ does the same job. But this transformation leaves $\hat{\lambda}$ exactly invariant![15]

We conclude that bound states of p- and q-7-branes cannot transform according to Eq. (5.13), and it is easy to see that they do not transform contravariantly either. Thus, they cannot transform as doublets.

It is evident that D-7-branes are not singlets. Thus, the next possibility to be tested is that 7-branes are triplets, i.e. they transform in the adjoint representation. This possibility looks particularly promising if we stick to the characterization of 7-brane bound states through monodromy matrices, which transform in the adjoint representation. Furthermore, there is no $SL(2,\mathbb{Z})$ transformation taking us from the monodromy matrix of the $(p=1)$ state, T, to the monodromy matrix of the $(p=-1)$ state, T^{-1}.

To clarify completely this issue we are going to make a precise definition of the charges involved and their relation with the monodromy matrix. First, we observe that the equations of motion for the scalars can be written as (we suppress hats here):

$$\nabla_\mu \mathcal{J}^\mu = 0\,, \qquad \mathcal{J}_\mu = 2\partial_\mu \mathcal{M}\mathcal{M}^{-1} = 2\begin{pmatrix} \frac{1}{2} j^{(\varphi)}_\mu & j_\mu \\ & \\ j^{(0)}_\mu & -\frac{1}{2} j^{(\varphi)}_\mu \end{pmatrix}, \tag{5.17}$$

where

[15] As we said before, the group acting on $\hat{\lambda}$ is $PSL(2,\mathbb{Z}) \equiv SL(2,\mathbb{Z})/\{\pm\mathbb{I}_{2\times 2}\}$.

$$\left\{\begin{array}{rcl} j^{(\varphi)}_{\mu} & = & e^{2\varphi}\partial_{\mu}|\lambda|^2\,, \\ & & \\ j^{(0)}_{\mu} & = & e^{2\varphi}\partial_{\mu}C^{(0)}\,, \\ & & \\ j_{\mu} & = & -C^{(0)}j^{(\varphi)}_{\mu} + |\lambda|^2 j^{(0)}_{\mu}\,. \end{array}\right. \tag{5.18}$$

The divergences of the first two currents are the dilaton and RR scalar equations of motion. The divergence of the third current is zero on shell but it is not an equation of motion. These three conserved currents can be associated to the three parameters of $SL(2,\mathbb{R})$. In fact, the Noether current associated to the global $SL(2,\mathbb{R})$ transformation $\Lambda = e^m$ where m is the mass matrix defined in Eq. (2.30) is given by

$$j^{(m)}_{\mu} = \mathrm{Tr}\left(\mathcal{J}_{\mu} m\right)\,. \tag{5.19}$$

Using the current matrix we can define a conserved charge matrix

$$\mathcal{Q} \equiv \left(\begin{array}{cc} \frac{\delta}{2}r & \beta q \\ \gamma p & -\frac{\delta}{2}r \end{array}\right) \equiv \tfrac{1}{2}\oint_{S^1}\mathcal{J} = \oint_{S^1} d\mathcal{M}\mathcal{M}^{-1}\,, \tag{5.20}$$

where p,q,r are integer charges and δ,β,γ are the adequate normalization constants. r is the charge associated to the dilatation current:

$$2\alpha r = \oint j^{T_1} = 2\oint j^{(\varphi)}\,, \tag{5.21}$$

p is the charge associated to shifts of the RR scalar

$$2\gamma p = \oint j^{\frac{1}{2}(T_2+T_3)} = 2\oint j^{(0)}\,, \tag{5.22}$$

and therefore the D-7-brane charge, and q is the charge associated to the remaining independent transformation

$$2\beta q = \oint j^{\frac{1}{2}(T_2-T_3)} = 2\oint j\,. \tag{5.23}$$

Observe that both the current matrix and charge matrix transform in the adjoint representation under $SL(2,\mathbb{R})$. Using the scalar equations of motion as we have written them in Eq. (5.17) it is possible to dualize the scalars on-shell and substitute the current matrix $\mathcal{J}^{\mu}$ by the Hodge dual of a 9-form field-strength matrix. This matrix will also transform in the adjoint representation[16].

Let the S^1 be parametrized by $\xi \in [0,1]$: We define

$$\mathcal{Q}(\xi) \equiv \int_0^{\xi} d\mathcal{M}\mathcal{M}^{-1}\,, \quad \Rightarrow \quad d\mathcal{Q}(\xi) = d\mathcal{M}\mathcal{M}^{-1}\,. \tag{5.24}$$

[16]See *Note Added in Proof.*

If $\mathcal{Q}(\xi) = \mathcal{Q}\xi$, the differential equation can be integrated giving

$$\mathcal{M}(\xi) = e^{\frac{1}{2}\mathcal{Q}\xi}\mathcal{M}_0\, e^{\frac{1}{2}\mathcal{Q}^T\xi}\,, \tag{5.25}$$

so that the corresponding monodromy matrix reads

$$M_{(p,q,r)} = e^{\frac{1}{2}\mathcal{Q}}\,. \tag{5.26}$$

The restriction to $M \in SL(2,\mathbb{Z})$ implies the quantization of the charges (p, q, r). In particular it implies that there are no allowed *quantum* states with $p = q = 0, r \neq 0$. This seems to restrict the number of independent charge to just two: p and q. But it is not easy to talk about the number of independent integers related by a Diophantic equation: Not any pair p, q is allowed.

The general form Eq. (2.39) for an $SL(2,\mathbb{Z})$ matrix is useful to illustrate our result. Let us take the case $n = 1$. The other three integers n^i are a Pythagorean triplet and can be parametrized by three integers t, s, l with the only restriction that s and l are coprime and one of them is an even number:

$$n^1 = \pm t(s^2 - l^2)\,, \quad n^2 = \pm 2tsl\,, \quad n^3 = \pm t(s^2 + l^2)\,. \tag{5.27}$$

This restricted case already produces a monodromy matrix much more general than the M_{pq} in Eq. (5.16). Only two of the integers are independent and the three of them can be put in one-to-one correspondence with the charges p and q.

In any case, the important lesson at this stage is that given the monodromy matrix of a certain 7-brane configuration, the above relation immediately allows us to find the 7-brane charges.

To finish this Section, let us stress that these solutions are just examples of the general class of negative-charge 7-brane-type solutions that we write below in the Einstein frame:

$$\left\{ \begin{array}{rcl} ds^2_E & = & dt^2 - d\vec{y}_7^{\,2} - H_7 d\omega d\overline{\omega}\,, \\ \\ H_7 & = & |h|^2 \Im\mathrm{m}\hat{\lambda}\,, \\ \\ \partial_{\overline{\omega}}\hat{\lambda} & = & \partial_{\overline{\omega}} h = 0\,. \end{array} \right. \tag{5.28}$$

The holomorphic function h is nothing but a holomorphic coordinate change. The solutions with positive charge can be obtained by the transformation in Eq. (5.9).

5.1.1 Q-7-Branes

We can now generate the S duals of the D-7-brane. The rules found above allow us to identify their charges. However, we need a formulation in terms of 8-form potentials to understand physically whether r represents an independent 7-brane charge or not. We will present such a formulation elsewhere.

First, we will construct the Q-7-brane.

Any $SL(2,\mathbb{Z})$ transformation can be written as a product of S and T transformations

$$S = \eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{5.29}$$

raised to positive or negative powers. Under these transformations, the charges p, q, r transform as follows:

$$\left\{ \begin{array}{lcl} r & \stackrel{S}{\rightarrow} & -r\,, \\ \\ q & \stackrel{S}{\rightarrow} & -\gamma/\beta p\,, \\ \\ p & \stackrel{S}{\rightarrow} & -\beta/\gamma q\,, \end{array} \right. \qquad \left\{ \begin{array}{lcl} r & \stackrel{T}{\rightarrow} & r + 2\gamma/\delta p\,, \\ \\ q & \stackrel{T}{\rightarrow} & q - \delta/\beta r - \gamma/\beta p\,, \\ \\ p & \stackrel{T}{\rightarrow} & p\,. \end{array} \right. \tag{5.30}$$

We see that, as expected, from a configuration with only p charge (a D-7-brane) an S transformation generates a configuration with only q charge. We call the object described by this kind of solution a “Q-7-brane” and, taking the D-7-brane solution in Eq. (5.10) we can immediately find its form:

$$\begin{array}{c} Q7 \\ (7,0,2) \end{array} \quad \left\{ \begin{array}{rcl} d\hat{s}^2_{IIB} & = & (H^2_{D7} + A^2)^{1/2} \left[H_{D7}^{-1/2} \left(\eta_{ij} dy^i dy^j - dy^2 \right) - H_{D7}^{1/2} d\omega d\overline{\omega} \right], \\ \\ \hat{\lambda} & = & -1/(-A + iH_{D7})\,, \end{array} \right. \tag{5.31}$$

where

$$H_{D7} = \tfrac{1}{4\pi} \log \omega\overline{\omega}\,, \qquad A = \tfrac{1}{4\pi} i \log \omega/\overline{\omega}\,. \tag{5.32}$$

The T transformation generates out of the D-7-brane a configuration with a different constant value for $\hat{C}^{(0)}$. Although this is the only difference with the original D-7-brane solution, this constant value induces q-charge through the Witten effect. The presence of both p and q charges induces r-charge which here seems not to be independent.

5.2 7-Branes with a Compact Transverse Dimension

We want to transform 7-branes under T duality and therefore we need to consider the corresponding solutions with a compact transverse dimension.

If one of the transverse coordinates, say $x^1_2 \equiv y$ is compact $y \sim y + 2\pi\ell$ then the function H_{D7} that solves Eq. (5.2) in $\mathbb{R} \times S^1$ takes a different form (we set to zero the additive constant for simplicity):

$$H_{D7} = \frac{h_{D_7}}{2\ell} |x^2_2| + h_{D7} \log \sqrt{1 - 2e^{-|x^2_2|} \cos y/\ell + e^{-2|x^2_2|}}\,. \tag{5.33}$$

Usually, only the zero-mode in the Fourier expansion of this function is considered when performing T duality transformations because the only T duality rules known (Buscher’s

[3]) apply only to solutions independent of the compact coordinate (at least the metric has to be). This is a strong limitation which only recently started to be appreciated [39]. Nevertheless, the behavior of this zero-mode seems to be well understood and we will focus on it. In our case, then, we will take, for a single D-7-brane ($h_{D7} = 1/(2\pi e^{-\hat{\varphi}_0})$) and for $\ell = 1/2\pi$ ($y \sim y + 1$)

$$H_{D7} = \frac{h_{D_7}}{2\ell}|x_2^2|\,. \tag{5.34}$$

Restricting ourselves to the region $x_2^2 > 0$ for simplicity we find for the complex scalar $\hat{\lambda}$ the expression

$$\left\{ \begin{array}{rcl} \hat{\lambda}_{(p=-1)_0} & = & \frac{1}{2}\omega\,. \\ & & \\ \hat{\lambda}_{(p=+1)_0} & = & -\frac{1}{2}\overline{\omega}\,. \end{array} \right. \qquad \omega = y + ix_2^2 \tag{5.35}$$

Somewhat surprisingly, the solution does depend on the compact coordinate y. The metric does not, but, after an $SL(2,\mathbb{R})$ transformation, the string metric will depend on y while the Einstein metric will not.

Again, it is convenient to rewrite $\hat{\lambda}$ as follows:

$$\hat{\lambda}_{(p=1)_0} = \tfrac{1}{2}e^{-\hat{\varphi}_0}\left(\begin{array}{c} z \\ -\bar{z} \end{array}\right), \qquad z = y + ix_2^2\,. \tag{5.36}$$

Let us now start by analyzing the monodromy of the positive charge solution zeromode (the holomorphic one). The above function is regular everywhere: The D-7-brane has been smeared out. The only non-trivial cycle to study is the one along y, and one finds that the zeromode is shifted by 1/2. This is not an $SL(2,\mathbb{Z})$ transformation. To understand this result it is convenient to map the cylinder into the Riemann sphere with two punctures by means of the conformal transformation $1/w = e^{2\pi i z}$. w is the coordinate in the patch around infinity. Going around the origin in the w plane is the same as going around the cylinder's S^1 parametrized by y in the negative sense. The complex scalar zeromode becomes

$$\hat{\lambda}_{(p=1)_0} = -\tfrac{1}{2}\frac{1}{2\pi i}\log w\,, \tag{5.37}$$

which obviously corresponds to a D-7-brane with charge $-1/2$ placed at infinity in the Riemann sphere, i.e. at infinity in the cylinder (we are considering only the positive x_2^2 part of the cylinder). Something analogous happens at minus infinity. Then, the presence of a D-7-brane on a cylinder induces the presence of other D-7-branes at infinity. The D-7-branes at infinity have to have integer charge and thus we can only place a D-7-brane of charge ($p = 2$) to have a consistent picture. The situation is depicted in Figure 3 The monodromies along the compact coordinate measure the 7-brane charges at infinity and are, therefore $SL(2,\mathbb{Z})$ matrices as discussed in the previous section (now with $\xi = y$). These are precisely the monodromy matrices that appear in our massive 9-dimensional type II supergravity theory.

Figure 3: If we place a 7-brane on a cylinder, one has to take into account that automatically 7-branes are created at the boundaries. This can be easily seen by conformally transforming the cylinder into a punctured sphere. Consistency of the monodromy implies that the total sum of the charges in the sphere is nil.

In the supergravity theory, the monodromy matrices are determined by the mass matrix m and, comparing with the results of the previous Section, this is identical to the pq-7-brane charge matrix m:

$$m = \tfrac{1}{2}\begin{pmatrix} m^1 & m^2 + m^3 \\ m^2 - m^3 & -m_1 \end{pmatrix} = \mathcal{Q} = \begin{pmatrix} \frac{\delta}{2}r & \beta q \\ \gamma p & -\frac{\delta}{2}r \end{pmatrix} . \tag{5.38}$$

This is the sought for relation between the background of 7-branes and the mass parameters of the massive 9-dimensional type II supergravity theory.

6 KK-7A- and KK-8A-branes and T Duality

In this Section we are going to check explicitly the dualities between extended objects underlying the generalized T duality between the type IIA and type IIB theories. We will find some of the objects whose existence we conjectured in the Introduction. We will essentially prove the connections shown in Figure 4.

It is convenient to start with the 11-dimensional Kaluza-Klein monopole which we refer to as KK-7M-brane. This is a 7-dimensional, purely gravitational object, but one of the spacelike worldvolume directions, with coordinate z is compactified on a circle. Its metric is given by

$$\begin{array}{c} KK7M \\ (6,1,3) \end{array} \left\{ \begin{array}{rcl} d\hat{\hat{s}}^2 & = & \eta_{ij}dy^i dy^j - H^{-1}\left(dz^2 + A_m dx^m\right)^2 - H d\vec{x}_3^{\,2}\,, \\ 2\partial_{[m}A_{n]} & = & \epsilon_{mnp}\partial_p H\,, \end{array} \right. \tag{6.1}$$

where $\vec{x}_3 = (x^m) = (x^1, x^2, x^3)$ and $i = 0, 1, \ldots, 6$. The standard solution corresponds to the choice

$$H = 1 + \frac{h}{|\vec{x}_3|}\,. \tag{6.2}$$

We can reduce this solution in three different ways. First, we can reduce in the isometry direction, z. It is well-known that the resulting object is the D-6-brane. Reducing on one of the standard spacelike worldvolume directions (double dimensional reduction) trivially gives the KK-6A-brane, which is nothing but the 10-dimensional KK monopole.

Finally, we can reduce it on a transverse coordinate, x^3. We obtain

$$\begin{array}{c} KK7A \\ (6,1,2) \end{array} \left\{ \begin{array}{rcl} d\hat{s}^2_{IIA} & = & \left(\frac{H}{H^2+A^2}\right)^{-1/2}\left[\eta_{ij}dy^i dy^j - \frac{H}{H^2+A^2}dz^2 - H d\omega d\overline{\omega}\right]\,, \\ e^{\hat{\phi}} & = & \left(\frac{H}{H^2+A^2}\right)^{-3/4}\,, \\ \hat{C}^{(1)}{}_{\underline{z}} & = & \frac{A}{H^2+A^2}\,, \\ \partial_\omega A & = & i\partial_\omega H\,, \end{array} \right. \tag{6.3}$$

where $\omega = x^1 + ix^2$ and $A = A_3$ and the last equation is simply $2\partial_{[m}A_{n]} = \epsilon_{mnp}\partial_p H$ with the assumption that H does not depend on x^3 and in the $A_1 = A_2 = 0$ gauge. In complex notation the last equation then reads $\partial_\omega\left(A_3 - iH\right) = 0$, which has as a particular solution

$$H = \tfrac{h}{2}\log\omega\overline{\omega}\,, \qquad A = \tfrac{h}{2}i\log\omega/\overline{\omega}\,. \tag{6.4}$$

This kind of solutions has been previously considered in Refs. [40, 41, 42]. To relate it with type IIB solutions, we further reduce it in the isometry direction z. The resulting solution is a 9-dimensional "Q-6-brane":

$$
\begin{array}{c} Q6_9 \\ (6,0,2) \end{array}
\left\{
\begin{array}{rcl}
ds^2_{II} & = & (H^2+A^2)^{1/2}\left[H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}d\omega d\overline{\omega}\right]\,, \\
e^{\phi} & = & \left(\frac{H}{H^2+A^2}\right)^{-1}\,, \\
C^{(0)} & = & \frac{A}{H^2+A^2}\,, \\
\partial_\omega A & = & i\partial_\omega H\,.
\end{array}
\right.
\tag{6.5}
$$

This is a solution of our massive 9-dimensional type II theory with $m^{i=0}$. We are going to show it through duality arguments.

Notice that we have obtained two different solutions by reducing first on z and then on x^3 and in the inverse order. The difference is a rotation in internal space z, x^3 and, by T duality to an S duality transformation in the type IIB side, as we are going to see.

We can now uplift this solution using the type IIB rules and adding the coordinate y. We obtain the Q-7-brane solution Eq. (5.31):

$$
\begin{array}{c} Q7 \\ (7,0,2) \end{array}
\left\{
\begin{array}{rcl}
d\hat{s}^2_{IIB} & = & (H^2+A^2)^{1/2}\left[H^{-1/2}\left(\eta_{ij}dy^i dy^j - dy^2\right) - H^{1/2}d\omega d\overline{\omega}\right]\,, \\
\hat{\lambda} & = & -1/(-A+iH)\,.
\end{array}
\right.
\tag{6.6}
$$

This solution is the S dual of the standard D-7-brane solution. In fact, performing the $SL(2,\mathbb{Z})$ transformation S and substituting the explicit expressions for H and A we get

$$
\begin{array}{c} D7 \\ (7,0,2) \end{array}
\left\{
\begin{array}{rcl}
d\hat{s}^2_{IIB} & = & H^{-1/2}\left(\eta_{ij}dy^i dy^j - dy^2\right) - H^{1/2}d\omega d\overline{\omega}\,, \\
\hat{\lambda} & = & -\frac{h}{i}\log\overline{\omega}\,,
\end{array}
\right.
\tag{6.7}
$$

which is the (positive charge) D-7-brane solution of Eq. (5.10) if we set $h = 1/2\pi$.

We could have reduced the KK-7A-brane on another transverse direction x^2. Equivalently, we could have simultaneously reduced the KK-7M-brane on x^2 and x^3. We immediately face a problem: if H is a harmonic function that only depends on x^1, then $A_1 = 0$ but A_2 and/or A_3 depend on x^3 and/or x^2.

The situation is identical to that of the reduction of the Q-7-brane on a transverse coordinate. There, it was impossible to eliminate the dependence on that coordinate and generalized dimensional reduction was necessary. Here, only through generalized dimensional reduction of 11-dimensional supergravity one can find the 9-dimensional solution and the T dual. The T dual must have a special isometric direction and 7 standard spacelike worldvolume coordinates. Such a configuration is what we call a KK-8B-brane. The generalized dimensional reduction of 11-dimensional supergravity must give the same 9-dimensional theory as the reduction of type IIB in presence of KK-8B-branes.

We could have reduced the KK-7A-brane on a standard worldvolume direction y^i, getting

$$KK7_9 \atop (6,1,1) \quad \left\{ \begin{array}{rcl} ds^2_{II} & = & H\left[H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2\right] - H^{-3/2}dz^2\,, \\ \\ e^{\phi} & = & H^{1/8}\,, \\ \\ k & = & H^{1/4}\,. \end{array} \right. \tag{6.8}$$

This *not* a solution of our 9-dimensional massive type IIB theory. It would be a solution of another massive 9-dimensional type II theory with "Killing vectors" in its Lagrangian. Only after the elimination by reduction of the special isometric direction will we get a solution to some massive supergravity. Anyway, if we uplift this configuration to ten dimensions using the standard type IIB rules we get

$$\begin{array}{c} Unknown \\ (6,2,1) \end{array} \quad d\hat{s}^2_{IIB} = H\left[H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2\right] - H^{-1/2}dy^2 - H^{-3/2}dz^2\,. \tag{6.9}$$

This purely gravitational configuration is similar to the KK-9M-brane but with 2 isometric directions instead of just one. Its presence as a 10-dimensional type IIB background will give an 8-dimensional fully covariant massive type II theory.

Objects of this kind can be useful in considering massive theories in lower dimensions, which are out of the scope of this paper and so we will not discuss them any further.

We have already checked the left hand side of Figure 4. It is convenient now to start from the KK-9M-brane, recently constructed and studied in Ref. [21][17]. This purely gravitational field configuration is not a solution of the standard 11-dimensional supergravity, but it is a solution of the massive 11-dimensional supergravity constructed in Ref. [18] which we have just generalized in a manifestly $SL(2,\mathbb{R})$-covariant way. Its defining property is that it has a special isometric direction (z) and reduction in this direction gives the D-8-brane.

Choosing $\epsilon = -1$, the metric of the KK-9M-brane is

$$\begin{array}{c} KK9M \\ (8,1,1) \end{array} \quad \left\{ \begin{array}{rcl} d\hat{\hat{s}}^2 & = & H^{1/3}\eta_{ij}dy^i dy^j - H^{-5/3}dz^2 - H^{4/3}dx^2\,, \\ \\ H & = & c + Qx\,, \end{array} \right. \tag{6.10}$$

where now $i = 0, 1, \ldots, 8$.

If we reduce the KK-9M-brane in the isometry direction (z) we get the D-8-brane

$$\begin{array}{c} D8 \\ (8,0,1) \end{array} \quad \left\{ \begin{array}{rcl} d\hat{s}^2_{IIA} & = & H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2\,, \\ \\ e^{\hat{\phi}} & = & H^{-5/4}\,, \end{array} \right. \tag{6.11}$$

which is a solution of Romans' massive type IIA supergravity [9].

Reducing further in one of the spacelike worldvolume directions (y^8) we get the 9-dimensional D-7-brane

[17] In that reference it is called "M-9-brane". We prefer the name KK-9M-brane because it stresses the fact that it has a special isometric direction as the usual KK monopole.

$$
\begin{array}{c} D7_9 \\ (7,0,2) \end{array}
\left\{
\begin{array}{rcl}
ds^2_{II} & = & H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2\,, \\
\\
e^{\phi} & = & H^{-9/8}\,, \\
\\
k & = & H^{-1/4}\,.
\end{array}
\right.
\tag{6.12}
$$

Uplifting to 10 dimensional using the type IIA rules we get the D-7-brane is also the solution we obtained by compactifying in a transverse dimension the D-7-brane. This establishes T duality between the D-8- and the D-7-brane [8].

If we reduce first the KK-9M-brane on a standard worldvolume direction we get the following field configuration

$$
\begin{array}{c} KK8A \\ (7,1,1) \end{array}
\left\{
\begin{array}{rcl}
d\hat{s}^2_{IIA} & = & H\left[H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2\right] - H^{-3/2}dz^2\,, \\
\\
e^{\hat{\phi}} & = & H^{1/4}\,,
\end{array}
\right.
\tag{6.13}
$$

which we call KK-8A-brane. This is not a solution of any standard 10-dimensional supergravity. Instead, it is a solution of the massive type IIA supergravity that one finds by reduction of the massive 11-dimensional supergravity of Ref. [18] in a direction different from the isometric one. This theory is related by a rotation in internal space with Romans' massive supergravity [9].

Reducing further in the isometry direction (z), we get the 9-dimensional Q-7-brane

$$
\begin{array}{c} Q7_9 \\ (7,0,1) \end{array}
\left\{
\begin{array}{rcl}
ds^2_{II} & = & H\left[H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2\right]\,, \\
\\
e^{\phi} & = & H^{5/8}\,, \\
\\
k & = & H^{-3/4}\,.
\end{array}
\right.
\tag{6.14}
$$

We observe again that we have obtained two different 9-dimensional results which must be related by a rotation in the 2-dimensional internal space and, therefore, by an S duality transformation in the T dual type IIB theory. Thus, not surprisingly, if we uplift the 9-dimensional Q-7-brane to ten dimensions using the standard type IIB rules we get

$$
\begin{array}{c} Q7^{\rm bare} \\ (7,0,2) \end{array}
\left\{
\begin{array}{rcl}
d\hat{s}^{2\ \rm b}_{IIB} & = & H\left[H^{-1/2}\left(\eta_{ij}dy^i dy^j - dy^2\right) - H^{1/2}d\omega d\overline{\omega}\right]\,, \\
\\
\hat{\lambda}^{\rm b} & = & +iH^{-1}\,.
\end{array}
\right.
\tag{6.15}
$$

This is nothing but the *bare* field configuration of the Q-7-brane Eq. (6.6). Using the generalized rules for uplifting, the dependence on the internal coordinate is fully recovered. This establishes T duality between the Q-7-brane and the KK-8A-brane under the generalized Buscher T duality rules of Appendix C.

7 Conclusion

We have successfully completed the program put forward in the Introduction. However, there are still some missing pieces in the general picture. In particular, we have found explicitly the T duals of the S duals of the D-7-brane (the KK-8A-brane, which comes from the reduction of the KK-9M-brane recently constructed in Ref. [21]) but we have not built the corresponding 9-form potential, related by T duality to the 8-form potential of the Q-7-brane which we have not constructed either.

If the picture we have proposed is correct and the 9-form potential of the KK-8A-brane is a purely gravitational object (as suggested by Hull in Ref. [43]) there might an analogous solutions in each string theory. Thus, there ought to be KK-8-branes of the type IIB theory (KK-8B-branes) and we can ask ourselves what the effect of having one of these objects in the background would be. Clearly, we could obtain a massive 9-dimensional type II theory with one more mass parameter! To what theory would it correspond in the type IIA/M side?

The answer is simple: If we reduce the massive 11-dimensional theory of Section 4.1 directly to nine dimensions we can use Scherk & Schwarz's original generalized dimensional reduction to produce an extra 9-dimensional mass parameter. As we have seen, the reduction of the 11-dimensional KK monopole (KK-7M-brane) over two transverse dimensions can only be performed using this technique. Eventually one could construct a 9-dimensional type II theory with four mass parameters. It is reasonable to expect that they are organized in a multiplet of the 9-dimensional duality group ($GL(,2\mathbb{R})$.

However, we only know how to produce one mass parameter using this technique if we compactify at least two dimensions. Thus, we do not know what the 10-dimensional theory would be like.

The next question would be to ask what would happen if we placed more KK-9M-branes in eleven dimensions or KK-8-branes in ten (or KK-($d-2$)-branes in d dimensions). Clearly this should give us new massive theories in dimensions lower than nine and should be seen in the T dual picture as the result of a generalized dimensional reduction (i.e. putting Q-($d-3$)-branes in the background).

The general picture we have obtained seems to agree with the suggestion of Ref. [44] of the existence of a general massive theory of which all other should be particular cases. In fact, the general massive 4-dimensional type II theory (massive $N = 8$ supergravity) has to be consistent with U duality, which acts on the mass parameters. These should then fit into a multiplet of E_7. On the other hand, since the mass parameters are in a sense potentials associated to branes, the maximally massive $N = 8$ supergravity should be considered *the* $N = 8$ supergravity theory and *the* low-energy limit of type II string theories. The presence of KK-($d-2$)-branes in the corresponding higher-dimensional theories which lead to the maximally massive $N = 8$ supergravity is unavoidable.

In Figure 5 we represent the present knowledge about classical solutions of 11- and 10-dimensional supergravity theories describing string/M-theory solitons. Missing from this *bestiarium* are still objects such as orientifolds which we do not know as classical solutions.

Finally, we would also like to comment on the possible relation between the superalgebras of supergravity theories in presence of KK-$(d-2)$-branes and the superalgebras studied by Bars in twelve and higher dimensions [34]. These superalgebras can accommodate naturally vectors that break Poincaré covariance. Being global algebras, a proof is not easy, but one is nevertheless tempted to identify those vectors with the Killing vectors of our supergravity theories. The presence of a preferred vector would therefore be the signal of the presence of KK-$(d-2)$-branes in the background.

Note Added in Proof

Soon after the appearence of this paper, another paper [45] appeared which, using other techniques, reached the same conclusions as the ones presented in Sec. (5).

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A 9-Dimensional Einstein Fields Vs. 10-Dimensional Type IIA String Fields

In the main body of the paper we went directly from 11 to 9 dimensions and thus we need to repeat the reduction from 11 to 10 dimensions [1, 6] to be able to relate 9- with 10-dimensional fields.

As usual, we assume now that all fields are independent of the spacelike coordinate $z = x^{\underline{10}}$ and we rewrite the fields and action in a ten-dimensional form. The dimensional reduction of 11-dimensional supergravity Eq. (4.4) gives rise to the fields of the ten-dimensional $N = 2A, d = 10$ supergravity theory

$$\left\{\hat{g}_{\hat{\mu}\hat{\nu}}, \hat{B}_{\hat{\mu}\hat{\nu}}, \hat{\phi}, \hat{C}^{(3)}{}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \hat{C}^{(1)}{}_{\hat{\mu}}, \right\} . \tag{A.1}$$

The metric, the two-form and the dilaton are NS-NS fields and the three-form and the vector are RR fields. We are going to use for RR forms the conventions proposed in Refs. [25, 26, 18].

The fields of the 11-dimensional theory can be expressed in terms of the 10-dimensional ones as follows:

$$
\begin{aligned}
\hat{\hat{g}}_{\hat{\mu}\hat{\nu}} &= e^{-\frac{2}{3}\hat{\phi}}\hat{g}_{\hat{\mu}\hat{\nu}} - e^{\frac{4}{3}\hat{\phi}}\hat{C}^{(1)}{}_{\hat{\mu}}\hat{C}^{(1)}{}_{\hat{\nu}}\,, & \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{C}^{(3)}{}_{\hat{\mu}\hat{\nu}\hat{\rho}}\,, \\
\hat{\hat{g}}_{\hat{\mu}\underline{z}} &= -e^{\frac{4}{3}\hat{\phi}}\hat{C}^{(1)}{}_{\hat{\mu}}\,, & \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\underline{z}} &= \hat{B}_{\hat{\mu}\hat{\nu}}\,, \\
\hat{\hat{g}}_{\underline{zz}} &= -e^{\frac{4}{3}\hat{\phi}}\,.
\end{aligned}
\tag{A.2}
$$

For the Elfbeins we have

$$
\begin{aligned}
\left(\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}}\right) &= \begin{pmatrix} e^{-\frac{1}{3}\hat{\phi}}\hat{e}_{\hat{\mu}}{}^{\hat{a}} & e^{\frac{2}{3}\hat{\phi}}\hat{C}^{(1)}{}_{\hat{\mu}} \\ 0 & e^{\frac{2}{3}\hat{\phi}} \end{pmatrix}, \\
\left(\hat{\hat{e}}_{\hat{\hat{a}}}{}^{\hat{\hat{\mu}}}\right) &= \begin{pmatrix} e^{\frac{1}{3}\hat{\phi}}\hat{e}_{\hat{a}}{}^{\hat{\mu}} & -e^{\frac{1}{3}\hat{\phi}}\hat{C}^{(1)}{}_{\hat{a}} \\ 0 & e^{-\frac{2}{3}\hat{\phi}} \end{pmatrix}.
\end{aligned}
\tag{A.3}
$$

Conversely, the 10-dimensional fields can be expressed in terms of the 11-dimensional ones by:

$$
\begin{aligned}
\hat{g}_{\hat{\mu}\hat{\nu}} &= \left(-\hat{\hat{g}}_{\underline{zz}}\right)^{\frac{1}{2}}\left(\hat{\hat{g}}_{\hat{\mu}\hat{\nu}} - \hat{\hat{g}}_{\hat{\mu}\underline{z}}\hat{\hat{g}}_{\hat{\nu}\underline{z}}/\hat{\hat{g}}_{\underline{zz}}\right), & \hat{C}^{(3)}{}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}}\,, \\
\hat{C}^{(1)}{}_{\hat{\mu}} &= \hat{\hat{g}}_{\hat{\mu}\underline{z}}/\hat{\hat{g}}_{\underline{zz}}\,, & \hat{B}_{\hat{\mu}\hat{\nu}} &= \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\underline{z}}\,, \\
\hat{\phi} &= \tfrac{3}{4}\log\left(-\hat{\hat{g}}_{\underline{zz}}\right).
\end{aligned}
\tag{A.4}
$$

After some standard calculations that we omit we find the bosonic part of the $N = 2A, d = 10$ supergravity action in ten dimensions in the string frame:

$$
\begin{aligned}
\hat{S} = \int d^{10}x\,\sqrt{|\hat{g}|}\,\Big\{ & e^{-2\hat{\phi}}\left[\hat{R} - 4\left(\partial\hat{\phi}\right)^2 + \tfrac{1}{2\cdot 3!}\hat{H}^2\right] \\
& - \left[\tfrac{1}{4}\left(\hat{G}^{(2)}\right)^2 + \tfrac{1}{2\cdot 4!}\left(\hat{G}^{(4)}\right)^2\right] - \tfrac{1}{144}\tfrac{1}{\sqrt{|\hat{g}|}}\,\hat{\epsilon}\partial\hat{C}^{(3)}\partial\hat{C}^{(3)}\hat{B}\Big\}\,.
\end{aligned}
\tag{A.5}
$$

where the fields strengths are defined as follows:

$$
\left\{
\begin{aligned}
\hat{H} &= 3\partial\hat{B}\,, \\
\hat{G}^{(2)} &= 2\partial\hat{C}^{(1)}\,, \\
\hat{G}^{(4)} &= 4\left(\partial\hat{C}^{(3)} - 3\partial\hat{B}\hat{C}^{(1)}\right),
\end{aligned}
\right.
\tag{A.6}
$$

and they are invariant under the gauge transformations

$$
\left\{
\begin{array}{rcl}
\delta\hat{B} & = & \partial\hat{\Lambda}\,, \\[2ex]
\delta\hat{C}^{(1)} & = & \partial\hat{\Lambda}^{(0)}\,, \\[2ex]
\delta\hat{C}^{(3)} & = & 3\partial\hat{\Lambda}^{(2)} + 3\hat{B}\partial\hat{\Lambda}^{(0)}\,.
\end{array}
\right.
\tag{A.7}
$$

Now, using these results together with the relation between 9- and 11-dimensional fields obtained in Section 4.1 we get

$$
\begin{array}{rcl}
\mathcal{M} & = & e^{\hat{\phi}}|\hat{g}_{\underline{xx}}|^{-1/2}\begin{pmatrix} e^{-2\hat{\phi}}|\hat{g}_{\underline{xx}}| + (\hat{C}^{(1)}{}_{\underline{x}})^2 & \hat{C}^{(1)}{}_{\underline{x}} \\[2ex] \hat{C}^{(1)}{}_{\underline{x}} & 1 \end{pmatrix}\,, \\[4ex]
K & = & e^{\hat{\phi}/3}|\hat{g}_{\underline{xx}}|^{1/2}\,,
\end{array}
\tag{A.8}
$$

$$
\begin{array}{rcl}
A_{(1)\ \mu} & = & \hat{B}_{\mu\underline{x}}\,, \\[2ex]
\vec{A}_{(1)\ \mu} & = & \begin{pmatrix} \hat{C}^{(1)}{}_{\mu} - \hat{C}^{(1)}{}_{\underline{x}}\hat{g}_{\mu\underline{x}}/\hat{g}_{\underline{xx}} \\[2ex] -\hat{g}_{\mu\underline{x}}/\hat{g}_{\underline{xx}} \end{pmatrix}\,, \\[4ex]
\vec{A}_{(2)\ \mu\nu} & = & \begin{pmatrix} \hat{C}^{(3)}{}_{\mu\nu\underline{x}} - 2\hat{B}_{[\mu|\underline{x}|}\hat{C}^{(1)}{}_{\nu]} + 2\hat{C}^{(1)}{}_{\underline{x}}\hat{B}_{[\mu|\underline{x}|}\hat{g}_{\nu]\underline{x}}/\hat{g}_{\underline{xx}} \\[2ex] \hat{B}_{\mu\nu} + 2\hat{B}_{[\mu|\underline{x}|}\hat{g}_{\nu]\underline{x}}/\hat{g}_{\underline{xx}} \end{pmatrix}\,, \\[4ex]
A_{(3)\ \mu\nu\rho} & = & \hat{C}^{(3)}{}_{\mu\nu\rho} - \tfrac{3}{2}\hat{g}_{[\mu|\underline{x}|}\hat{C}^{(3)}{}_{\nu\rho]\underline{x}}/\hat{g}_{\underline{xx}} - \tfrac{3}{2}\hat{C}^{(1)}{}_{\underline{x}}\hat{g}_{[\mu|\underline{x}|}\hat{B}_{\nu\rho}/\hat{g}_{\underline{xx}} \\[2ex]
 & & -\tfrac{3}{2}\hat{C}^{(1)}{}_{[\mu}\hat{B}_{\nu\rho]}\,, \\[2ex]
g_{E\ \mu\nu} & = & e^{-4\hat{\phi}/7}|\hat{g}_{\underline{xx}}|^{1/7}\left[\hat{g}_{\mu\nu} - \hat{g}_{\mu\underline{x}}\hat{g}_{\nu\underline{x}}/\hat{g}_{\underline{xx}}\right]\,.
\end{array}
\tag{A.9}
$$

B 9-Dimensional Einstein Fields Vs. 10-Dimensional Type IIB String Fields

Using the results of Section 2 we find

$$
\begin{aligned}
\mathcal{M} &= \Lambda^{-1}(y)\hat{\mathcal{M}}(\hat{x})(\Lambda^{-1})^T(y) = \hat{\mathcal{M}}^{\mathrm{b}} = e^{\hat{\varphi}^{\mathrm{b}}} \begin{pmatrix} |\hat{\lambda}^{\mathrm{b}}|^2 & \hat{C}^{\mathrm{b}\ (0)} \\ \hat{C}^{\mathrm{b}\ (0)} & 1 \end{pmatrix} , \\
K &= e^{\hat{\varphi}/3}|\hat{\jmath}_{\underline{yy}}|^{-2/3} = e^{\hat{\varphi}^{\mathrm{b}}/3}|\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}}|^{-2/3} , \\
A_{(1)\ \mu} &= \hat{\jmath}_{\mu\underline{y}}/\hat{\jmath}_{\underline{yy}} = \hat{\jmath}^{\mathrm{b}}{}_{\mu\underline{y}}/\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}} , \\
\vec{A}_{(1)\ \mu} &= -\Lambda^{-1}(y) \begin{pmatrix} \hat{C}^{(2)}{}_{\mu\underline{y}} \\ \hat{\mathcal{B}}_{\mu\underline{y}} \end{pmatrix} = \begin{pmatrix} \hat{C}^{\mathrm{b}\ (2)}{}_{\mu\underline{y}} \\ \hat{\mathcal{B}}^{\mathrm{b}}{}_{\mu\underline{y}} \end{pmatrix} , \\
\vec{A}_{(2)\ \mu\nu} &= \Lambda^{-1}(y) \begin{pmatrix} \hat{C}^{(2)}{}_{\mu\nu} \\ \hat{\mathcal{B}}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \hat{C}^{\mathrm{b}\ (2)}{}_{\mu\nu} \\ \hat{\mathcal{B}}^{\mathrm{b}}{}_{\mu\nu} \end{pmatrix} , \\
A_{(3)\ \mu\nu\rho} &= -\hat{C}^{(4)}{}_{\mu\nu\rho\underline{y}} - \tfrac{3}{2}\hat{\mathcal{B}}_{[\mu\nu}\hat{C}^{(2)}{}_{\rho]\underline{y}} - \tfrac{3}{2}\hat{\mathcal{B}}_{[\mu|\underline{y}|}\hat{C}^{(2)}{}_{\nu\rho]} , \\
g_{E\ \mu\nu} &= e^{-4\hat{\varphi}/7}|\hat{\jmath}_{\underline{yy}}|^{1/7}\left[\hat{\jmath}_{\mu\nu} - \hat{\jmath}_{\mu\underline{y}}\hat{\jmath}_{\nu\underline{y}}/\hat{\jmath}_{\underline{yy}}\right] .
\end{aligned}
\tag{B.1}
$$

C Generalized Buscher T Duality Rules

Now we just have to compare the results of Appendix B and Appendix A to identify the 10-dimensional fields of the type IIA and IIB theories. This identification produces for us the searched for generalization of Buscher's T duality rules [3]. These rules generalize the standard type II T duality rules of Ref. [6] in the same way as those of Ref. [8]: The rules have exactly the same form as the massless ones if we replace the type IIB fields by the *bare type IIB* fields.

The only deficiency of these rules is with respect to the S duals of D-7-branes: It is necessary to dualize their 8-form potentials which transform independently of $\hat{\lambda}^{\mathrm{b}}$.

Thus, indicating by a superscript b the bare type IIB fields the T duality rules take the form[18]:

From IIA to IIB:

[18] These rules apply to RR n-forms for any n. For the values of n that do not appear in the main body of this paper, one simply has to use the general expression for the RR field strengths and gauge transformations given in Ref. [18] inspired by those of Refs. [25, 26].

$$
\begin{aligned}
\hat{\jmath}^{\rm b}{}_{\mu\nu} &= \hat{g}_{\mu\nu} - \left(\hat{g}_{\mu\underline{x}}\hat{g}_{\nu\underline{x}} - \hat{B}_{\mu\underline{x}}\hat{B}_{\nu\underline{x}}\right)/\hat{g}_{\underline{xx}}\,, & \hat{\jmath}^{\rm b}{}_{\mu\underline{y}} &= \hat{B}_{\mu\underline{x}}/\hat{g}_{\underline{xx}}\,,\\
\hat{\mathcal{B}}^{\rm b}{}_{\mu\nu} &= \hat{B}_{\mu\nu} + 2\hat{g}_{[\mu|\underline{x}}\hat{B}_{\nu]\underline{x}}/\hat{g}_{\underline{xx}}\,, & \hat{\mathcal{B}}^{\rm b}{}_{\mu\underline{y}} &= \hat{g}_{\mu\underline{x}}/\hat{g}_{\underline{xx}}\,,\\
\hat{\varphi}^{\rm b} &= \hat{\phi} - \tfrac{1}{2}\log|\hat{g}_{\underline{xx}}|\,, & \hat{\jmath}^{\rm b}{}_{\underline{yy}} &= 1/\hat{g}_{\underline{xx}}\,,
\end{aligned}
$$

$$
\begin{aligned}
\hat{C}^{{\rm b}\;(2n)}{}_{\mu_1\ldots\mu_{2n}} &= \hat{C}^{(2n+1)}{}_{\mu_1\ldots\mu_{2n}\underline{x}} + 2n\hat{B}_{[\mu_1|\underline{x}|}\hat{C}^{(2n-1)}{}_{\mu_2\ldots\mu_{2n}]}\\
&\quad -2n(2n-1)\hat{B}_{[\mu_1|\underline{x}|}\hat{g}_{\mu_2|\underline{x}|}\hat{C}^{(2n-1)}{}_{\mu_3\ldots\mu_{2n}]\underline{x}}/\hat{g}_{\underline{xx}}\,,\\
\hat{C}^{{\rm b}\;(2n)}{}_{\mu_1\ldots\mu_{2n-1}\underline{y}} &= -\hat{C}^{(2n-1)}{}_{\mu_1\ldots\mu_{2n-1}}\\
&\quad +(2n-1)\hat{g}_{[\mu_1|\underline{x}|}\hat{C}^{(2n-1)}{}_{\mu_2\ldots\mu_{2n-1}]\underline{x}}/\hat{g}_{\underline{xx}}\,.
\end{aligned}
\tag{C.1}
$$

From IIB to IIA:

$$
\begin{aligned}
\hat{g}_{\mu\nu} &= \hat{\jmath}^{\rm b}{}_{\mu\nu} - \left(\hat{\jmath}^{\rm b}{}_{\mu\underline{y}}\hat{\jmath}^{\rm b}{}_{\nu\underline{y}} - \hat{\mathcal{B}}^{\rm b}{}_{\mu\underline{y}}\hat{\mathcal{B}}^{\rm b}{}_{\nu\underline{y}}\right)/\hat{\jmath}^{\rm b}{}_{\underline{yy}}\,, & \hat{g}_{\mu\underline{x}} &= \hat{\mathcal{B}}^{\rm b}{}_{\mu\underline{y}}/\hat{\jmath}^{\rm b}{}_{\underline{yy}}\,,\\
\hat{B}_{\mu\nu} &= \hat{\mathcal{B}}^{\rm b}{}_{\mu\nu} + 2\hat{\jmath}^{\rm b}{}_{[\mu|\underline{y}|}\hat{\mathcal{B}}^{\rm b}{}_{\nu]\underline{y}}/\hat{\jmath}^{\rm b}{}_{\underline{yy}}\,, & \hat{B}_{\mu\underline{x}} &= \hat{\jmath}^{\rm b}{}_{\mu\underline{y}}/\hat{\jmath}^{\rm b}{}_{\underline{yy}}\,,\\
\hat{\phi} &= \hat{\varphi}^{\rm b} - \tfrac{1}{2}\log|\hat{\jmath}^{\rm b}{}_{\underline{yy}}|\,, & \hat{g}_{\underline{xx}} &= 1/\hat{\jmath}^{\rm b}{}_{\underline{yy}}\,,
\end{aligned}
$$

$$
\begin{aligned}
\hat{C}^{(2n+1)}{}_{\mu_1\ldots\mu_{2n+1}} &= -\hat{C}^{{\rm b}\;(2n+2)}{}_{\mu_1\ldots\mu_{2n+1}\underline{y}}\\
&\quad +(2n+1)\hat{\mathcal{B}}^{\rm b}{}_{[\mu_1|\underline{y}|}\hat{C}^{{\rm b}\;(2n)}{}_{\mu_2\ldots\mu_{2n+1}]}\\
&\quad -2n(2n+1)\hat{\mathcal{B}}^{\rm b}{}_{[\mu_1|\underline{y}|}\hat{\jmath}^{\rm b}{}_{\mu_2|\underline{y}}\hat{C}^{{\rm b}\;(2n)}{}_{\mu_3\ldots\mu_{2n+1}]\underline{y}}/\hat{\jmath}^{\rm b}{}_{\underline{yy}}\,,\\
\hat{C}^{(2n+1)}{}_{\mu_1\ldots\mu_{2n}\underline{x}} &= \hat{C}^{{\rm b}\;(2n)}{}_{\mu_1\ldots\mu_{2n}}\\
&\quad +2n\hat{\jmath}^{\rm b}{}_{[\mu_1|\underline{y}|}\hat{C}^{{\rm b}\;(2n)}{}_{\mu_2\ldots\mu_{2n}]\underline{y}}/\hat{\jmath}^{\rm b}{}_{\underline{yy}}\,.
\end{aligned}
\tag{C.2}
$$

The relation between the *bare* fields and the real fields is given in Section 2.

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Figure 4: This figure is a magnified and more detailed piece of Figure 5 in which a general picture of all the known extended objects of M/string theory and their duality relations is given. Only well-established relations are shown, and so no duality connections between the conjectured KK-8B-brane and other objects are drawn. In the triplets (m, n, p) m stands for the number of standard spacelike dimensions of the object, n for the number of special isometric directions (z) and p for the number of standard transverse dimensions. The double arrows indicate on which directions T duality acts.

Figure 5: Duality relations between classical solutions of 10- and 11-dimensional supergravity theories describing string/M-theory solitons: p-branes, M-branes, D-branes, gravitational waves, Kaluza-Klein monopoles and other KK-type solutions. Lines with two arrows denote T duality relations. Dashed lines denote S duality relations. Lines with a single arrow denote relations of dimensional reduction, either vertical (direct dimensional reduction) or diagonal (double dimensional reduction).